
****** All Kinds of Mathematicks, Theoretical and Practical, are Taught by Mr. *Harris*, at his House in *Amen-Corner*: Where also Gentlemen may be Boarded.

 I think my self oblig'd in Justice to Recommend Mr. *John Rowley*, Mathematical Instrument-maker under St. *Dunstan's Church* in *Fleet-Street*, to all Lovers of Art and Judges of Good Work, as the Best Workman I know, and of whom all sorts of Mathematical Instruments may be had at Reasonable Rates.

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Short, but yet Plain

ELEMENTS OF GEOMETRY AND PLAIN TRIGONOMETRY.

S H E W I N G

How by a Brief and Easie Method, most of what is Necessary and Useful in *Euclide*, *Archimedes*, *Apollonius* and other Excellent *Geometricians*, both Ancient and Modern, may be Understood.

Written in *French* by *F. Ignat. Gaston Pardies*.

And rendred into *English*,
By JOHN HARRIS M. A. and F. R. S.

The *Second Edition*, in which are many New Propositions, Additions and useful Improvements: The Problems being now placed every where in their proper Order, and the whole Accommodated to the Capacities of young Beginners.

London, Printed by *J. Matthews*, for *R. Knaplock* at the *Angel*, and *D. Midwinter* and *T. Leigh*, at the *Rose* and *Crown* in *St. Paul's Church-Yard*. 1702.



TO

My Worthy Friend
Charles Cox Esquire,
Member of Parliament for the
Burgh of Southwark.

Dear SIR,

Among the many Obligations you have
conferr'd upon me, I account it not
the least, that you have given me a
Rise to revive my Mathematical Studies;
in which as I have formerly spent some time,
so I know of no more Useful way of employ-
ing my leisure Hours.

And indeed, Sir, the Diversion and Advan-
tage I have lately reaped from them hath
(by the Divine Blessing) both supported me
under, and in a good Measure carried me
through such Pressures and Difficulties as I
once almost despair'd of Surmounting.

A 3

The

The Epistle Dedicatory.

The Mathematick Lecture which you at first set up Gratis in your Burgh, and which out of an uncommon Generosity, you have Since removed into the City of London, is a Demonstrative Proof both of your sincere Endeavour to promote the good of your Country, and also of your Capacity to do it the best way. And as I have already in part, so I hope to see such Effects from so Noble a Design, as will render your Name justly Honourable to Posterity as well as to this present Age. Sir, you know your Self and Me too well, to take this for Flattery. 'Tis what Truth, Justice and Gratitude Oblige me to say.

I shall only add, that I am again glad of this opportunity to shew the Just Esteem I have of your Merit, and the equal Regard I have for your Freinds^{hip}. I am,

S I R,

Your most Obliged

Humble Servant,

John Harris.

THE
TRANSLATOR
TO THE
READER.

After frequent Perusal, and mature Deliberation on this Book; I judge it to be the plainest, shortest, and yet easiest Geometry I have ever seen Published: And therefore I thought it very well worth my while to let it appear again in our own Language, as it had already done twice before in the Latin Tongue. And 'tis so well esteemed of, by very Competent Judges amongst Us, as to be read in our Universities, by Tutors to their Pupils: and also, which is not usual with Books of this kind, there hath been an entire Impression sold off in a little more than a Years Time.

As to the Translation; I have by no means Obligated my self servilely to follow the the French way of Expression; for indeed a Literal Version of a Book out of any Language will be scarce Intelligible in English. I have therefore all along aimed rather to give you F. Pardies Sense, than his Words; and have

The Translator, &c.

have made him speak what I judge he would have done, had he wrote in our Language. I have made no scruple to add any thing that I saw necessary to render him clear and intelligible; and particularly in this Second Edition I have added many things, which were not in the Former; (as that of the Quadrature of any assigned Segment of a Lune &c.): and have put the Problems every where in their natural Order.

Wherever I found a Demonstration which wanted (as I thought) a better Method or a clearer Turn, I have endeavoured to give it one: where I Judged it of no great use, I have now quite omitted it, as I have done also the Author's Preface, that the Price of the Book might not be encreased by the new Additions. And I hope it will fully Answer the Author's Design in Writing it and mine in Translating it, viz. to render this most Noble and most Useful Science, easily attainable by young Beginners: which it hath done so effectually, as to gain by it the ill word of one or two of our Teachers about Town; who hate a Plain Book written by another, as much as they dread to get a Schollar of their own of a good Capacity: the reason of which is very obvious.

Advice

Advice to those who would Understand Geometry.

1. They ought to enure themselves to consider well the *Figures*, at the same time as they Read the *Propositions*. There will be some Labour and difficulty at first, but they will break thro' it in two or three Days.

2. They ought not to be discouraged, if they meet with some things which they do not understand at first; Geometry is not so easily to be attain'd, as History.

3. If, after they have Read and Consider'd attentively any Proposition, they find they don't Understand it; let it be pass'd over, it will probably be Intelligible by reading farther, or at least when they have gone over the whole, and have began to Read it over anew. There are indeed many things in Geometry, that will never be well understood at first Reading over.

4. The Numbers which are within the Parentheses, *v. gr.* (3. 14). shew that the Matter there spoken of hath been proved elsewhere, *viz.* in this Instance, in the fourteenth Article of the *Third Book*; And they ought always to mind the Number of the Article, and

(a)

to

Advice to those, &c.

to consult the Places referr'd to, that so the
may gain the Demonstration of what the
Read.

5. When they meet with any Words which
they don't Understand, they must consult the
Table at the End of the Book.

6. 'Tis good to have a Master at first, to
Explain to them the Nature and Manner of the
Demonstrations: For by that means they will
Understand the thing both much easier and
much sooner, than they can do by Reading
by themselves.

E L E.

ELEMENTS OF GEOMETRY.

BOOK I.

Of Lines and Angles.

BY the Word *Quantity*, we mean that which being compared with another thing of the same Nature, may be said to be Greater or Less than, Equal or Unequal to it. As Extension, Number, Weight, Time, Motion. And all those things which are capable of being so compared as to More or Less, are the Object of Geometry.

2. We design nevertheless to consider now only Extension; as being that which serves for an *Example* and *Rule* to Measure all other Quantities by.

3. That Quantity which being supposed without any Breadth or Thickness, is extended only in Length, is called *A Line*. That which hath both length and Breadth, (but is supposed to have no Thickness) is called a *Surface* or *Superficies*: And that which hath Length, Breadth and Thickness, is called a *Body* or a *Solid*.

2 ELEMENTS

4. *A Point* is that which is considered as having a manner of Dimensions; and as being Indivisible in every Respect. The Ends or Extremities of Lines, also the middle of them, are Points.

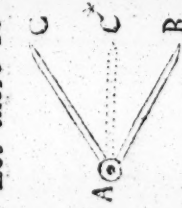
5. There are *Strait Lines*, and there are *Crooked Curved Ones*: Also there are Even and Plain Surfaces which are called *plan's*; and there are *Crooked or Curved Ones*: Which like a Vault, (or the Tilt of a Boat or Wagon) are *Convex* above, and *Concave* below.

6. When two Lines meet in a Point, The *Aperture Distance* or *Inclination* between them is called an *Angle*. Which when the Lines forming it are Right *Strait Ones*, is called a *Rectilineal Angle*, as A. But when they are *Crooked*, 'tis called a *Curvilinear One*, as B. And when one is *Strait* and the other *Crooked*, 'tis called a *Mixt Angle*, as C.

N. B. The Lines forming any Angle are Called its Legs.



7. That *Angle* is said to be *less* than another, whose Legs are more inclined to, (or *nearer to*) each other. Let there be Two Lines A B and A C meeting in the



Point A. If you imagine those Lines to be moveable like the Legs of a pair of Compasses, and yet fastned together on A, as with a Joint; 'Tis ealie then to conceive, that the farther they are Opened or Parted from one another the *greater* will be the Angle between them: As on the Contrary, the nearer they are brought together the more will they incline towards each other, and so the Angle between them must be the *less*.

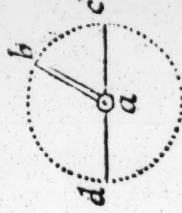
Book I. of GEOMETRY. 3

8. It must therefore be observed that the *Quantity of Angles* is by no means measur'd by the *Length* of their Legs, but by their *Inclination*. Thus, *v. gr.* the Angle B is bigger than A; tho' the Legs of the Angle B, are much shorter than those of A: But then those of A are much more inclined to each other, than those of B. And to apprehend this the Better, Imagine the Angle B to be put upon A; as you may conceive by the Prickt Lines about A, which represent the Legs of B lying on it. For 'tis plain the Angle A will be easily contained within B; and that its Legs are much more inclined to one another, than those of B, and therefore it is less than B.



9. An Angle is usually marked by 3 Letters, of which the middlemost, and which always is placed at the Angular Point where the Lines meet, denotes the Angle. As in the Figure following *b a c* denotes the Angle made by the two Lines *b a* and *c a* meeting in the Point *a*.

10. If we Imagine the Line *a b* fastned by its End *a* in the middle of the Line *d c* but yet so as to be moveable on *a* as on a Centre: If then you conceive it be moved quite round till it arrive at the place where it began, the Point *b* will describe a Curve Line, which is usually called a Circle; but 'tis rather the *Circumference of a Circle*; for properly speaking, the Circle is the *Space* contain'd within the Circumference.



11. Any part of the Circumference is called an Ark, as *b c*.

12. The Line *d c* (passing through the Centre) and terminated by the Circumference is called the *Diameter*, and divides the Circle into two equal Parts. Also every right Line passing thro' the Centre *a* (and Terminated at each end by the Circumference) divides the Circle into two equal Parts, and will be a Diameter.

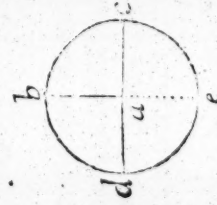
13. The

B 2

13. The Line ab or ac , or any other drawn from the Center to the Circumference, is called the Radius or Semidiameter.

14. All *Radii* or *Semidiameters* (of the same or equal Circles,) are equal (*as is plain from the Genesis of a Circle given in Art. 10.*)

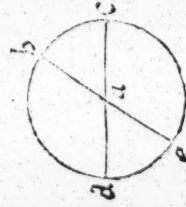
15. When the End b of the Radius ab is equally distant from the two Ends of the Diameter dc ; That



(with dc) make 4 Right Angles; and it will be another Diameter; which with the former dc will divide the Circle into 4 equal Parts.

16. Then those Lines are said to be *Perpendicular* one to another: viz. ba to dc , and da to be .

17. But if b be nearer to one End of the Diameter



(or Right Line) dc than it is to the other, it is then said to fall on the other *Obliquely*; and it makes with dc two Angles that are *Unequal*: Of which the Lesser bac is called *Acute*, and the Greater dab is called *Obtuse*.

If the Line ab be produced to e , it will be a new Diameter, and will make below two other Angles: So that in the whole there will be four Angles; of which those two that touch only in the Angular Point, as bac and ead , as also dab and eac are called *Vertical* or *Opposite* Angles. But those that have one Leg common to both, as dab and bac ; and bac and eac , are called *Adjoining* or *contiguous* Angles.

18. Those

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5

18. Those Angles which (at Equal distances from the Angular Point) are subtended by Equal Arks, are also equal themselves. As if the Ark bc be proved Equal to the Ark de , then will the Angle bac be Equal to dac .

19. The two *Contiguous Angles* taken together, are always equal to two Right ones.

For as the Line de is a Diameter, and therefore cuts the Circle into two Equal Parts, the two Arks ab and bc taken together, will be equal to a Semicircle. Wherefore the two Angles dab and bac together will be equal to two Right ones, because they compleat the whole Semicircle, as two Right ones do.

(Art. 15.)

20. So that this Proposition is of Universal Truth, That one Right-Line falling on another makes the *Contiguous Angles (together) equal to two Right ones*. For if the Lines are *Perpendicular* to each

other as pa is to dc , then 'tis plain the Angles must be Right (by the 15.) And if the Line fall *Obliquely* as ba doth: then indeed the Angles are Unequal; But as much as the Obtuse one dab exceeds one Right Angle, by so much is the Acute one bac exceeded by the other Right one. So that the Smallness of one is compensated by the Greatness of the other.

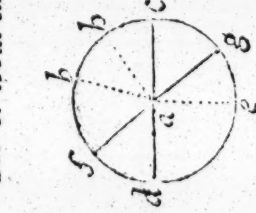


21. If two Angles which have one Leg Common to both, do make Angles equal to two Right ones, their other Legs do make but one Right Line. Let the Angle dab and bac be (together) equal to two Right ones. Then I say that the Lines da and ac do join together, as to make one Right Line (*vid. Fig. in Art. 17.*) which is clear from what hath been said. For if on the Center a you describe a Circle $dace$, the two Arks db and bc will be equal to a Semi-Circle, because the two Angles dab and bac are supposed to be equal to two Right ones. Wherefore the Lines da and

B 3

and ca will make a Diameter, and consequently be joined into one Right Line.

22. If from the Point a you draw several Lines, as



ad, af, ab, ah, ag, ac . they will makes diverse Angles; and all those Angles taken together, be they more or less, will be equal to four Right ones. For 'tis clear all these Angles together do compleat the Circle $dbee$, whose Circumference they divide into as many Arks as there are Angles. Now all these

together are equal to four Quarters of a Circle; which is as much as to say that all the Angles are equal in the whole to four Right ones; for so many Right Angles do also compleat the Circle.

AXIOM I.

If to or from Equal Things you add or subtract Equals, the Sums or Remainders will be equal.

23. The Vertical or Opposite Angles are Equal. Let there be two Right Lines dac and bac (crossing or cutting one another in the Point



a .) I say the Angle d ae is Equal to b ac . For the Ark db with the Ark bc makes a Semicircle; and so doth the same Ark bd with the Ark de . Therefore the Ark bc must be equal to de ; because the Ark

db continues the same, whether it help to compleat the Semicircle with de or bc : (wherefore being taken away from both, it must leave the Ark d e $=$ b c . But if the Arks be equal, the Angles subtended by them must be so too, and therefore the Angle d ae is Equal to b ac .) and by the same Reason the Angle d ab will be Equal to e ac .

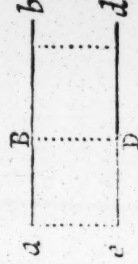
Book I. of GEOMETRY.

7

24. The Circumference of every Circle is (*Supposed to be*) divided into 360 equal Parts, which are called Degrees: And every Degree into 60 Minutes, every Minute into 60 Seconds, every Second into 60 Thirds, and so on Infinitely. And to determine the Quantity of any Angle, we compute the Degree that (*the Ark which is its Measure*) doth contain. *v. gr.* When we speak of an Angle of 90 deg. we mean a Right Angle; because the Right Angle contains the fourth Part of the whole Circumference, which is 90 deg. the fourth Part of 360. So an Angle of 60 deg. is an Angle that contains two Thirds of a Right one.

25. (*Degrees are marked either with Degr. or usually with a small Cypher over the last Figure, as 60°. Minutes with a small Line as 50', Seconds with two such, as 30''; Thirds with three such, as 25''' &c. So that 25° 32' 42'', is to be read, 25 Degrees, 32 Minutes, 43 Seconds.*)

26. Two Lines are said to be *Parallel*, when they run always equi-distant from each other. Thus the two Lines *ab* and *ed* are *Parallel*, if they are equally distant from each other in *ae*, in *BD*, in *bd*, and in all other Places.



27. This Distance is always measur'd by a Perpendicular; as if from the Point *a* you imagine the line *ae* to fall Perpendicular to *ed*; as also doth the Line *bd* on the same Line; we naturally conceive that if those two Perpendiculars are of the same Length, or equal; the two Lines *ab* and *ed* are equally distant from each other in those two places, which is self-evident and needs no Proof.

28. Two Parallel Lines being continued infinitely, yet can never meet. For being always equally distant, there may any where be drawn between them a Perpendicular equal to *ae* or *bd* and consequently they can never meet.

B 4

29. If

The

29. If a Line cut or cross two other Lines that are Parallel, it will be equally inclined to them both:



And if a Line cutting two other Lines, be equally inclined to them both; those two Lines are Parallel.

Let the two Parallel Lines be ae and df , which are cut by the Line gh and $gabh$. I say that that Line $gabh$, as b is equally inclined to ae as it is to df .

that is, the Angle gae is equal to the Angle ghd , which will be clear to any one that considers the thing: For if the Angle gae , for Instance, were greater than ghd , and if the Line ae were opened or parted farther from ag ; then must the Point e of the line ae incline or approach towards f ; because bf is supposed not to part or open from gb as ae doth from ga . Therefore the two Lines ae and df will not be Parallel, (which contradicts the Supposition:) 'Tis the same thing if the Angle gae should be said to be less than ghd , for then the Lines could not be Parallel.

Moreover, if we imagine the two Parallel Lines to be like the two Edges of a Ruler, that Ruler may be considered as one (*broad and*) indivisible Line.

So that the Angles bhd and gae may be considered as being contiguous Angles, and equal to two Right Angles (by 20:) And the Angles bhd and gae as Vertical or Opposite Angles (23:) and so equal to each other. But if bhd be equal to gae , its equal ghf (23.) must be so too. Therefore the Parallel Lines have the same inclination, or make equal Angles, with the crossing Line.

30. Whenever a Right-line cuts two Parallels it makes with them eight Angles: Of which four a, b, k, g , are External; and the other four c, d, e, f , are Internal. The Angles c and f , as also d and e are called Alternate. The Angles e and a as



also f and b are called the Internal and Opposite on the same side. And the Angles d and f , as also c and e are called the Internal Angles on the same side.

A X I O M II.

Things equal to a Third, are equal to one another.

31. The Alternate Angles, *c* and *f*, as also *d* and *e*; and the Internal and Opposite ones on the same side, as *b* and *f*, *a* and *e*, &c. are severally equal. (*The latter Part was proved in Art. 29. and the former follows plainly by the 22. for since b is equal to f, its Opposite Angle c must be so too, and if a be equal to e its Opposite d will be so also.* &c. (vid. Art. 29. the latter Part.

32. When a Line falls on two Parallel ones, it makes the Internal Angles on the same Side equal to Two Right ones.

I say the Angle *d* with *f*, is equal to two Right ones: Because *f* is equal to *c* (by 31.) and *c* and *d* together are equal to two Right ones (by 20.) Therefore *f* and *d* together must be equal to two Right ones, which was to be prov'd.

(*The same may may c and e together be proved Equal to two Right ones; for c and d taken together are so (by 20) but d is equal to e (31) Therefore c and e are equal to two Right ones.*)

33. One Proposition is called the *Converse* of another; when after a Conclusion is drawn from something *Supposed*, in the *Converse* Proposition that *Conclusion* is *Supposed*; and then that which was in the other *Supposed*, is now drawn as a Conclusion from it. For Example: We say here, if two Lines are Parallel, (*and another Cross them*,) the Angles *d* and *f* together are Equal to two Right ones: Where we suppose the Lines to be Parallel, and from thence conclude those Angles must be Equal to two Right ones: But the *Converse* is Thus; If the *Internal Angles on the same side*, *d* and *f* together, are Equal to two Right ones: then those Lines are Parallel: Where after we have supposed these Angles equal to two Right ones, we conclude the Lines are to be Parallel.

34. Con-

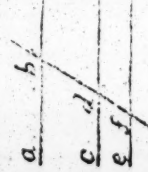
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34. Converse Propositions in this Case are very true; as, that if a Line cut two other Lines, and makes the Alternate Angles equal; Those two Lines are Parallel: which I desire the Reader to remember.

35. If two Lines are Parallel to a third Line, they are so to one another.

Let the Line ab be Parallel to cd ; and let ef also be Parallel to the same Line cd ; If say ab is Parallel to ef : For if you draw a Line as bdf cutting them all Three; the Angle b will be equal to d (by 31.) and the same d will be equal to f (by 31.) because ef is also Parallel to cd . Wherefore the Angle

b must be equal to f : Because by Axiom 2, if two things are equal to a third, they are so to one another. But if the Angle b be equal to f , then the Line ab is Parallel to ef (by 34.)



BOOK II.

Of Triangles.

1.

A Figure is a Space compassed round on all Sides. And if the Lines which Terminate it are all Right ones 'tis called a *Rectilinear* (or Right Lined) Figure: If they are Crooked 'tis called a *Curvilinear*; and if they are partly Right Lines and partly Crooked, 'tis called a *Mixt* Figure.

2. There are *Plane* Figures, which are *Plane Surfaces*, and there are *Solid* ones, which have three Dimensions. But we speak here only of *Plane Surfaces*, or *Plane Figures*.

3. All

Book II. of GEOMETRY. II

3. All the Lines which encompass any Figure taken together, make that which is called the Circumference, Perimeter, or the Compass of the Figure.

4. Of all Curvilinear or Mixt Plane Figures, in Common Geometry we consider properly only the Circle, or a Part of a Circle terminated on one side by an Ark, and on the other by one or more Right Lines.

Of Rectilinal Figures the most Simple are *Triangles* which are Terminated by three Right Lines (*and no more*) making as many Angles.

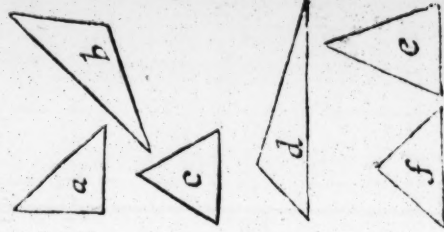
5. A Triangle as *a*, which hath one Right Angle, is a *Right-angled Triangle*; if it have one Angle Obtuse, 'tis called an *Obtuse-angled One*, as *b*; and if all its three Angles are Acute, 'tis called an *Acute-Angled Triangle*, as *c*.

7. If a Triangle have all its three sides Unequal 'tis called a *Scalene* as *d*. If it hath two sides equal, 'tis called an *Isosceles* as *e*; And if all the three sides are equal, 'tis called an *Equilateral one*, as *f*.

8. When two sides of a Triangle are considered, they may be called its *Legs*, and the third side may then be called the *Base*. But any one side may be called the *Base*, (tho' we usually and most properly call that so, which lies Parallel to the Horizon, and which is next to us.)

9. In every Triangle, the three Angles taken together, are equal to two Right ones.

Let the Triangle be *abc*: I say, that the Angle *a* added to the Angle *c*, added to the Angle *b* (or the Sum of all three) are equal to two Right ones. For let *de* be drawn Parallel to the Base *ac*, then will those two Parallel Lines be cut by the Line *bc*; and consequently the Alternate Angles *c* and *d* will

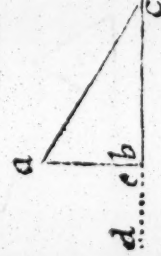


will be equal to each other (by 1.31.) Moreover the Line ba falling on, or cutting the same Parallels de and ac , will make the Internal and opposite Angles on the same side equal to two Right ones; that is, a added to abe are equal to two Right Angles (by 1.32.) But the Angle abe contains the two Angles b and d . So that the Angle a added to b added to d will be equal to two Right ones. But c being equal to d , it will follow that a added to b added to c ; or the Sum of all three together must be equal to two Right ones: which was to be Proved.



1c. If any side of a Triangle be produced, or drawn out, the external Angle will be equal to the two Internal Opposite Angles, (*taken together*). Let the Triangle be abc , whose Base cb draw out to d , by which means a New Angle as e will be made, which is called the *External Angle* of that Triangle. Then I say, that That External Angle e , is equal to both the Internal and Opposite ones a and c .

For those Angles a and c together with b are equal to two Right ones (by the Precedent,) and so also are e and b by (1.20.) wherefore e must be equal to a added to c , because together with b , it makes two Right Angles, as they do. Q. E. D.



C O R O L L A R I E S.

1. The Sum of the three Angles of all Triangles is the same.
2. No Triangle can have above one Right or Obtuse Angle.

3. If

Book II. of GEOMETRY. 13

3. If in any Triangle the Angle be Right, the other two must be Acute.

4. If in any Triangle there be one Angle equal to both the others, that must be a Right one.

5. If you know the degrees of one Angle in any Triangle, you know the Sum of the other two; for this what is wanting of 180° , and if the Sum of any two be known, the Quantity of the Remainder is known.

6. Hence if two Triangles have any two Angles respectively equal to one another, the remaining Angles must also be equal.

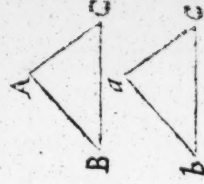
7. The Angle of an Equilateral Triangle is $\frac{1}{3}$ of 2 Rightangles or $\frac{2}{3}$ of 1 Rightangle.

8. Hence 'tis very easy to Trisect a Rightangle, by making on one of the Legs an Equilateral Triangle.

11. If a Triangle ABC hath two Sides AB and AC equal to two others ab and ac in another Triangle, and if also the Angle A be equal to a , I say the Third Side BC shall be equal to bc ; the Angle B equal to b , the Angle C to c , and the whole Triangle ABC to abc .

For if we imagine the Triangle abc to be placed upon ABC, so that the side ab shall lie exactly on its equal AB: Then must the side ac fall on its equal AC; because the Angle a is equal to A, and so the Point c will fall on C, and b upon B, and the whole Triangle abc on the Triangle ABC: because all things so exactly answer, that nothing of the upper Triangle, can fall besides the Under one.

12. Figures which do thus meet, fit, or answer to each other exactly, when they are placed one upon the other are



are called *Congruous Figures*, *Quia mutuo sibi congruunt*. And therefore the 4th Axiom is, *Quæ sibi mutuo congruunt sunt æqualia*; i. e. Those Figures which placed upon another do answer to and cover one another exactly, are equal.

13. It is also true, That, if a Triangle hath all 3 Sides equal to the 3 Sides of another Triangle, the Angles also in one, shall be equal to those in the other: And all the Space which one Triangle contains, shall be equal to that contained in the other.

As if AB be equal to ab , AC to a , and BC to bc : I say that the Angle A shall be equal a , B to b , and C to c , and the whole Triangle ABC , to abc , and this needs no other Proof.



14. If the Angle A be equal to a , the Angle B to b , and the Side AB to ab , Then shall the Side AC be always equal to ac , BC to bc ; and the whole Triangle ABC to abc : Which is eare to Prove by the preceding Propositions.

15. In every *Isosceles* Triangle, the Angles at the Base, opposite to the equal Legs, are equal.

Let the Triangle bab , whose Legs ab and ac are equal: I say, the Angle b is also equal to c . For Imagine the Base bc divided into two equal Parts in the Point d , then will the Line ad (*which has been drawn*) make of the whole, two Triangles, abd and adc , which will have all three Sides in one, equal to those in the other: For ab is equal to ac by the Supposition, and bd is equal to dc , and ad is common to both. Wherefore (by 2. 12.) the whole Triangle bab is equal to dad and the Angle b is equal to c , which was to be proved.



16. In an *Isosceles* Triangle, if a Line drawn from the Angle at the top do (*Bisect or*) divide the Base into two equal Parts, it is both Perpendicular to the Base, and also Bisefts the Angle at the Top into two equal Parts.

Parts. For (vid. Fig. præcedent.) the Angle adc is equal to the Angle adb (by the last) and consequently they must be both *Right ones*; and therefore the Line ad is Perpendicular to the Base bc (1.15.) and the Angle dac will be equal $da b$ (by the last Prop.)

17. In every Triangle the Greater side is always opposite to, or *subtends* to the Greater Angle.

In the Triangle abc let the Side bc be longer than ba ; then I say the Angle bac is subtended by the greater Side bc is bigger than the Angle c , which is subtended by the lesser Side. For let bd be taken equal to ba , then will abd be an *Isosceles* Triangle; whose Angle bda will be equal to dba (2.15.) But the Angle cab is bigger than bda ; the whole being greater than the Part) and therefore must be bigger than bda (which is equal to bda). Now the Angle adb is an External Angle in the Respect of the little Triangle adc ; and therefore must be bigger than the Internal one c (by 2. 10.) Wherefore the Angle bac being bigger than d , must certainly be bigger than c ; which was to be Proved.



18. Of all Lines that can be drawn from a Point given, the *Shortest* is the Perpendicular; and they are all other distant from it. Let the given Line be ad , and the Point given b ; let ba be Perpendicular to da ; let also bc and bd be drawn. I say that ba is the shortest Line that can possibly be drawn from b ; and (for instance) is shorter than bc (or any other that can be assigned) and I say also that bd is longer than bc .



For in the Triangle bac , the Angle a is a *Right one*, and consequently bigger than either of the other; because they must necessarily be both Acute (by the last Prop.) Therefore the side bc is longer than ba (2.17.) as subtending a greater Angle.

So

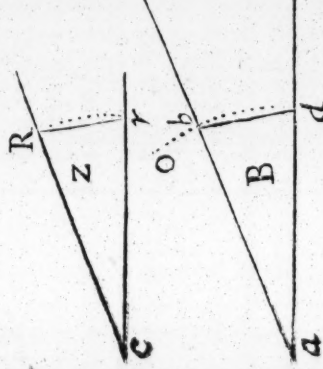
So also in the Triangle dbc , the Angle $dc b$ is *Obtuse*, because the Angle bca is *Acute*: And consequently the Side db must be longer than cb , as subtending a greater Angle (2. 17.)

19. In every Triangle any two Sides taken together are longer than the Third; because a Right line is the nearest Distance between any two Points.

PROBLEM I.

On a Line given ad , to make an Angle as B , equal to a given one Z .

Place the Compasses in c the Vertex of the given Angle, and describe the Ark Rr ; then keeping them at the same distance, set one Foot in a , one end of the given line, and with the other describe the Ark obd ; set Rr from d to b : and draw ab , so shall the Angle bad or B be equal to Z .



For the Legs of each are *Radii* of equal Circles, and the line bd was taken equal to Rr , wherefore the whole Triangles cRr and abd must be equal (by 13.) and consequently the Angle a equal to c .

PROBLEM II.

Hence the Practice of making all Sorts of Triangles, Equilateral, Isosceles; or with any given Angles or Sides, will easily appear.

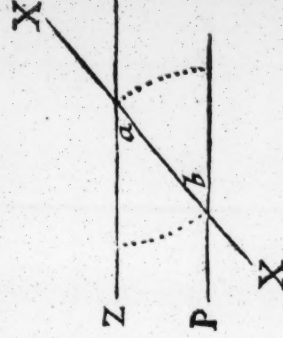
PROB-

PROBLEM III.

A Right Line, as P, being given, to draw thro' a, a Point given, the Line Za, Parallel to it.

Thro' *a* draw any Line, as *XX* making any Angle, as *b*, with the given Line: Then make the Angle $\angle Z a X = \text{to } b$, and *Za* shall be the Parallel sought.

For the alternate Angles *a* and *b* are equal by Construction; wherefore *Za* is Parallel to *Pb* (by 1.31) Q. E. D.

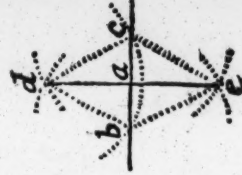


PROBLEM IV.

To bisect or divide a given Line cb into two equal Parts in the Point a.

Open the Compasses to more than the length of *abc*, and with that distance make at each end of *abc*, two pairs of intersecting Arks, as at *e* and *d*: then drawing the Line *ed*, it will bisect the given Line in *a*.

For the Triangles *bed* and *dec* are equal (by 2.13). Wherefore the Angles $\angle adb = \angle adc$. Therefore the Triangles *adb* and *adc* will be equal also (by 2.11) and consequently *ab* is equal to *ac*. Q. E. D.



C

PROB

B-

E-
or

PROBLEM V.

By much the same Method may a Perpendicular, as ad , be raised in the middle of any given Line, or one may be let fall from the Point e or d , to the given Line abc , And the Demonstration is the same in all.

And after the same way of Practice may the given Angle bdc be bisected.

If setting one Foot of the Compasses in d , you take db equal to dc : And then setting the compasses in b and c , strike the Arks Intersecting each other in e : So shall de bisect the Angle required.

BOOK III.

Of Quadrilateral Figures and Polygons.

1. **T**Hose Figures whose Sides are four Right Lines and those making four Angles, are called *Quadrilateral* or four-sided Figures.

2. When the opposite Sides are Parallel, the Quadrilateral Figure is called a *Parallelogram* as A ; but if not 'tis, called a *Trapezium* as B .



2. When

Book III. of GEOMETRY. 19

3. When the Parallelogram hath all its four Angles Right, 'tis called a *rectangled Parallelogram*; or for brevity sake a *Rectangle* as *c*:  

And if the Angles are right and the Sides are all equal, 'tis called a *Square*, as *d*.
4. If a *Parallelogram* hath all its Sides equal, but its Angles unequal, then 'tis called a *Rhombus*, as *e*.



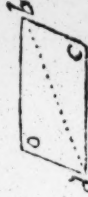
5. If a *Parallelogram* hath neither its Angles not sides all equal, 'tis called a *Rhomboides*, as *a*.

6. In every *Parallelogram* the opposite Angles are equal. Let the *Parallelogram* be *obcd*.



I say the Angle *o* is equal to *c*, for the Angle *o* is equal to the Alternate one *b* (1. 31) and the External one *b* is equal to the Internal one *c* (1. 31) wherefore *o* is equal to *c*.

7. A Line as *db* drawn across the Figure from Angle to Angle is called the *Diagonal*, and by some, the *Diameter*.



8. Every *Parallelogram* is divided into two *Equal Parts* by the *Diagonal*. The *Diagonal db* divides the *Parallelogram obcd* into the two equal Triangles *obd* and *bcd*. For, 1. The Angle *o* is equal to *c* (3. 6). 2. The Angle *obd* is equal to *cdb*. (1. 31) and for the same Reason, the Angle *obd* is equal to *cdb*; and the side *bd* is common to both these Triangles, wherefore the Triangle *obd* is equal to *cdb* (by 2. 14).

9. In every *Parallelogram*, the opposite Sides are always equal.

For, (Drawing the *Diagonal db*) the whole Triangle *dob* will be equal to the Triangle *bcd* by the foregoing Prop. And consequently the side *cd* must be equal to *ob* and the side *od*, to *cb*.

10. Two *Diagonals ac* and *bd* do bisect each other in the Middle at *e*.

For in the two Triangles aed and bec , The side ae is equal to be (3.9) The Angle ead is equal to ebc

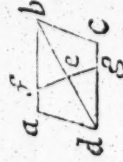
(1.31) and also the Angle ade is equal to bce (1.31) and moreover the (Vertical) Angles aed and bce are equal also (1.23) *Wherefore the whole Triangle aed is equal to the Triangle bce*



(2.14) and consequently the side de is equal to ec and the side ae to the side ec . The two Diagonals therefore bisect each other. Q. E. D.

11. Every Right Line as fg passing by the middle of a Diagonal divides the Parallelogram into two equal Parts.

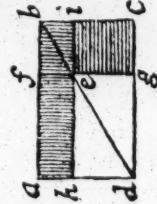
To demonstrate which, the *Trapezium* or Irregular Quadrilateral Figure $fgda$ must be proved equal to the *Trapezium* $fgcb$. And that is thus done. 1. The Triangle bef is equal to the Triangle $d eg$: For the



side de is equal to eb by the Supposition; and the Angle efb is equal to egd (1.31) and the opposite Angles ate are equal, wherefore the Triangle efb is equal to edg (2.14). 2. The

great Triangle abd is equal to bdc (3.8) wherefore if from the Triangle abd you take away the little Triangle feb , and instead of it put the Triangle edg (which is equal to feb) you will have the *Trapezium* $adeg$, which will be equal to the Triangle adb : This is, to just one half of the whole Parallelogram (3.8) which was to be proved.

12. If in the Diagonal db you take a Point as e , and thro' it draw two Lines hi and fg Parallel to the two



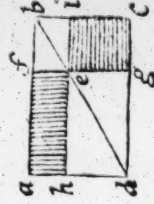
Sides of the Parallelogram, It will be divided by them into four little *Parallelograms*, i. e. $febi$, $begd$ (which two are called the *Parallelograms about the Diameter*) and $afhe$ and $egcd$; which other two are called the *Complements*. And those two Complements will be either of the Parallelograms about the Diameter made

Figure that is called a *Gnomon*. As you see in the Figure; where the *Gnomon* is distinguished by being shaded.

13. In every Parallelogram the Complements are Equal. We must prove that $ebaf$ is equal to $egci$.

DEMONSTRATION.

The whole Triangle abd is equal to the whole bdc (3.8) And the little Triangle efb is (for the same Reason) equal to ebi . And the Triangle bed is also (by the same) equal to edg . Wherefore if from the two equal Triangles abd and bdc we take away equal things, viz. if from one we take away efb and dbe , and from the other ebi and egd , There will remain on one side the Parallelogram $ebaf$, equal to the Parallelogram $ei cg$ which remains on the other, which was to be proved.



14. Parallelograms having the same Base, and being between the same Parallels are equal.

Let there be a Parallelogram $abcd$ and another efb , both on the same Base ab ; and let the Line cd when produced, be supposed to pass by ef ; So that the two Parallelograms shall be between the same Parallels, and Terminated by them; that is, between the two Parallels cf and ab . I say then that the Parallelogram $acdb$ is equal to $ae fb$.



For 1. cd is equal to ef , because both are equal to ab (3.9) Therefore if to each of those equal Lines you add the Line de , ce will be equal to df : 2. ca is equal to db (3.9) 3. The Angle ace is equal to bdf (1.31) wherefore the Triangle cae is equal to dbf . (2.11) Now if from each of these equal Triangles you take away the white Triangle doe that is between the Parallelograms, and add to them both the Triangle dob , there will arise the Parallelogram $edba$ on one side,

side, equal to the Parallelogram $ae fb$ on the other side. (And they must be equal, because when the white Triangle was taken away, there remained the Trapezium $caef$ equal to $ob fe$, and when at last to each of them was added the Triangle aob , the Sums must needs be equal. That is $cdab$ equal to $aefb$. Q. E. D.)

15. Parallelograms on equal Bases ab and $g b$, and between the same Parallels ab and cf , are equal.

For if we Imagine the third Parallelogram $ef ba$ to be drawn; that shall be equal to the Parallelogram

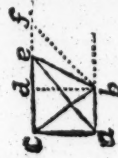
$ab cd$, because on the same Base



with it, and between the same Parallels ab and cf . And that Parallelogram will also be equal to $ef ba$

because it hath the same Base ef , with it (it matters not whether you reckon the Base above or below) and is between the same Parallels. Therefore $bfeg$ and $bdca$ being both equal to the third Parallelogram $ef ba$, must be equal to each other.

16. Triangles on the same Base ab , and being between the same Parallels cf and ab , are always equal.



The Triangle abc is equal to def

Because if you imagine a Line bd drawn Parallel to ac , and another as bf drawn Parallel to ae ; there will be made two Parallelograms $acdb$ and $ae fb$; which being on the same Base and between the same Parallels, will be equal to one another (3. 14).

But the Triangle abc is the half of the Parallelogram $acdb$, and the Triangle def is the half of the Parallelogram $ae fb$ (3. 8); wherefore, (since the wholes are equal, the halves must) and consequently the Triangle abc is equal to the Triangle def .

17. Triangles on equal Bases, and between the same Parallels, are also equal; as is very easy to prove from (3. 15).

18. If a Triangle have the same Base with a Parallelogram, and be also between the same Parallels, it shall be just the half of that Parallelogram. For it will still be equal to abc which is just half (3. 8) of $acdb$.

19. A *Pentagon* is a Figure having five Sides and five Angles.

If all the Sides are equal, and consequently the Angles, 'tis call'd a *Regular Pentagon*.

20. An *Hexagon* is a Figure of six Sides and Angles; an *Heptagon* of seven, an *Octagon* of eight, &c. which are all called *Regular* when they have equal Sides and Angles.

21. A *Polygon* in general signifies any Figure of many Sides and Angles; but no Figure is called by this Name, unless it have more than four or five Sides.

22. Every *Polygon* may be divided into as many Triangles as it hath Sides, if any where within the *Polygon* you take a Point as a , and from thence draw Lines to every Angle ab, ac, ad , &c. for they shall make as many Triangles as the Figure hath Sides.



23. The Angles of any *Polygon* taken all together, will make twice as many Right ones, except four, as the Figure hath Sides. *v.gr.* If the *Polygon* have six Sides; the double of that is 12, from whence take four, there remains eight. I say that all the Angles of that *Polygon*, viz. b, c, d, e, f, g , taken together, are equal to eight Right Angles. For the Lines ab, ac, ad , &c. do divide the Figure into six *Triangles*; the three Angles of each of which are equal to two right ones (2. 9) so that all their Angles together make 12 Right Ones. But now, each of these six *Triangles* hath one Angle in the Point a , and by it they complet the space all round the said Point. And all the Angles about that Point, are equal to four Right Ones (1. 22). Wherefore those four being taken from 12, (*The Summ of the Right Angles of all the six Triangles*) leaves eight, the Summ of the Right Angles of the *Hexagon*.

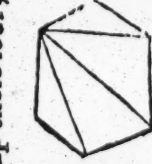
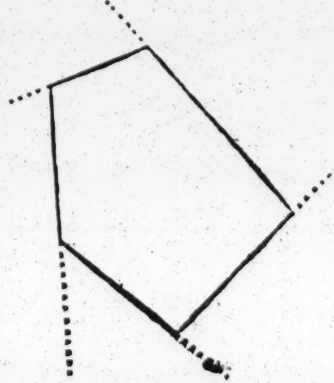
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So that the Figure hath plainly twice as many right Angles as it hath Sides, except four; which was to be proved.

C O R O L L A R Y.

All the External Angles of any Right-lined Figure, are equal to just four

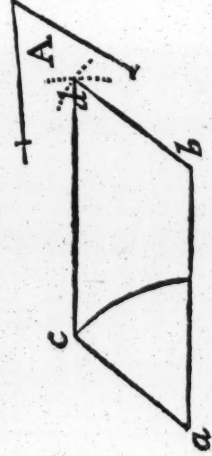
Right Ones: For drawing out the Sides as in the Figure, 'tis plain the Internal and External Angles together will make twice as many right Ones as the Figure hath Sides; but the Internal Angles are equal to all those except four (by this prop.) Wherefore the External Angles must make up these four, and no more.



24. A Polygon may be divided also into Triangles, by drawing Lines from Angle to Angle. But then the Number of the Sides will exceed that of the Triangles.

P R O B L E M I.

On a given Line $a b$, to make a Parallelogram, having an Angle equal to a given Angle A .



Make the Angle $\angle ab c = A$. Then take $a b$ in your Compasses, and setting one Foot in c , strike an Ark

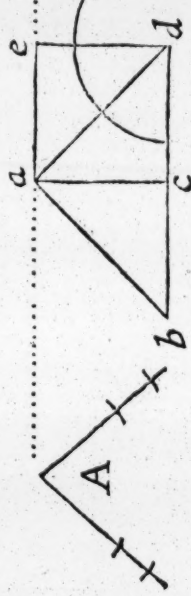
as d : next take the Distance ac , and placing one Foot in b , cross the Ark in d : draw cd and bd , and it is done.

And thus also may the Line cd be drawn Parallel to ab , thro' a Point assigned, and any Parallelogram readily be described.

PROBLEM II.

A Triangle abd being given, to make a Parallelogram equal to it, which shall have a given Angle $\equiv A$.

Bisect the Base of the Triangle in c : Make the Angle $cde \equiv A$, thro' the Vertex d draw ae Parallel to the



Base. Make $ae \equiv cd$ and draw ac . So will ce be the Parallelogram required.

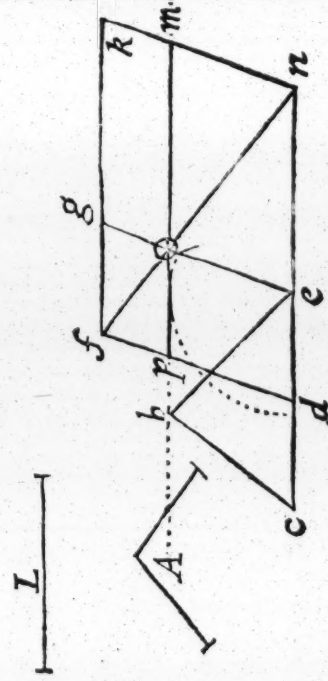
For being on but half the Base, and of the same height with the Triangle, it will be equal to it, by 18. of the third Book, and its Angle cde is equal to A . Q. E. F.

PROB.

PROBLEM III.

On a Line given as L, to make a Parallelogram, equal a given Triangle cbe, and having an Angle equal an Angle given, as A.

Make the Parallelogram do equal to the Triangle and having its Angle $e = A$, by Problem the last. Then produce po till om become equal to L , and draw out de till en be also equal to L . Then draw the Diagonal fn .



agonal no , producing it till it meet with dp , also produced to f . Then draw $fk = dn$, and $nk = df$; and That will compleat the Parallelogram fn , in which the Complement gm will be equal to pe (3. 13) which is equal to the Triangle cbe . Q. E. F.

And thus 'tis easie to make a Parallelogram equal to any Right-lined Figure given: By reducing that Figure into Triangles, &c.

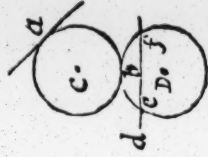
BOOK IV. Of A Circle.

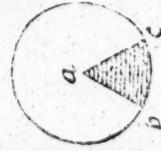
1. Line is said to *Touch* (or to be a *Tangent* to) a

Circle, when tho' produc'd both ways from the Point of Contact, it will only touch it, and not *cut* or enter within it. Thus the Line *a* Touches the Circle C. as that Circle C doth the Circle D; but *d* enters within the Circle, and *cuts* it; and is call'd a *Secant*.

2. If a right Lines enter within a Circle and cut it into two Parts, those Parts are called *Segments*: *b* is a less Segment and D a greater: That *part of the Line* cutting the Circle (and which is within it) is called a *Chord* as *e f*. And the Parts of the Circle (or rather *Circumference*) cut off, are called *Arks*: The Chord with the Ark make two mixt Angles as *e* and *f*, and they are call'd *Angles of a Segment*.

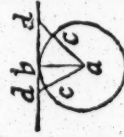
3. If you take a Point as *c*, in the Ark of any Segment, and from thence draw two Lines *ca* and *cb* (to the ends of the Chord) they shall make an Angle *acb*; which is call'd an *Angle in a Segment*: And that Angle *acb* is said to *insist* or *stand* on *abd*, the Ark of the other Segment below.





4. A Sector of a Circle is a mix'd Triangle comprehended between two Radii, ab , ac , and the Ark of the Circle bc , 'tis mark'd in the Figure by being shaded.

5. If at the end of any Radius, or Semidiameter, ab , you draw a Perpendicular, as db , it shall touch the Circle but in one Point. And all the Points of the Line bd shall be without the Circle. *v. g.* I say the Point d (or any other assignable) is without: For if you draw the Line ad from the Center, and *that* shall cut the Circle in the Point e , that Line ad will be longer than ab ; and consequently longer than ac which is equal to ab (1. 14). Wherefore the Point d is without the Circle. Q. E. D.

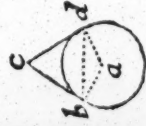


(2. 19) and consequently longer than ac which is equal to ab (1. 14). Wherefore the Point d is without the Circle. Q. E. D.

6. A Chord, as bc , is divided into two equal Parts (or Bisefted) by a Perpendicular da , drawn from the Center a . For the Triangle abc is an *Isoscelés*, because ba is equal to ca (1. 14) and therefore the Perpendicular ad Bisefts the Base bc (2. 16). The Ark bce is also by this means Bisefted.



7. Two Tangents, cb and cd , drawn from the same Point without a Circle, are equal one to another. For draw from the Center to the Points of Contact, ba and ad . Then will those Lines be Perpendiculars to the Tangents (by 4. 5) Then if you draw also the Line bd , the Angle abd will be equal to adb (2. 15). Wherefore if from the Right and (consequently) equal Angles cba and cda , you take away the equal ones abd and adb , the remaining Angles cbb and cdd will be equal: Wherefore their opposite sides must also be equal by the *converse* of (2. 15); That is, cb is equal to cd . Q. E. D.



cba and cda , you take away the equal ones abd and adb , the remaining Angles cbb and cdd will be equal: Wherefore their opposite sides must also be equal by the *converse* of (2. 15); That is, cb is equal to cd . Q. E. D.

8. Equal

8. Equal Chords as bc and fb , do cut off equal Segments bd and fg . And the Perpendiculars ae and di , drawn to them from the Center are also equal, as is easily proved; (saith *Pardie*, But he gives us no Demonstration.) Yet 'tis plainly thus prov'd: The Chords and Arks are both Bisected by the Perpendiculars



(4. 6) And therefore the Sectors cad , $da b$, $fa g$ and $ga b$, must be all equal; as also will all the Triangles x, z, o and k, be (2. 11). Therefore their Doubles will also be equal i. e. the Sector bac will be equal to $fa h$: And the Triangle bac to the Triangle $fa h$. And if these last Triangles are taken from the equal Sectors $ha f$ and bac , the Segments bdc and hgf must remain equal. That the Perpendiculars are equal, is plain from the Equalities of the Triangles z and o , or X and K .

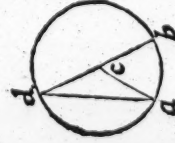
9. Let there be a Semidiameter Rc , and a Perpendicular (to it without the Circle) $R T$, another Line cutting the Circle in S , and a Perpendicular (let fall from thence) to the Radius $R C$ in n (a Point within the Circle). All these Lines have Artificial Names. The Line $T R$ is called the *Tangent* of the Ark $R S$ (which suppose 30° . $T C$ is call'd the *Sine* of the same Ark of 30° , and the Line $S n$ is call'd the (*Right*) *Sine* of the same Ark. $R C$ is by some called the *whole Sine*, but most usually the *Radius*.



10. If in the Circumference of a Circle, you take two Points as a and b , and from thence draw two Lines to the Center c , and two others to any Point, as d in the Circumference; they will make two Angles, of which acb is called an *Angle at the Center*, and adb an *Angle at the Circumference*.



11. The Angle at the Center acb is always double to one at the Circumference adb (*inscribing with it on the same Ark a b.*)

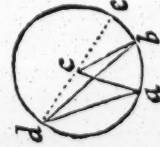


Of which there are three Cases.

1. If one of the Lines as db , pass thro' the Center c , then 'tis plain the external Angle acb (2. 10) will be equal to both the Internal and Opposite ones a and d taken together.

But the two Angles d and a are equal, because acd is an *Isosceles* Triangle, whose side ac is equal to cd (2. 15). Therefore the Angle c at the Center being equal to both, is double of either alone: That is double to d . Q. E. D.

2. If neither of the Lines db , da , (*which form the Angle at the Circumference*) pass thro' the Center c : (*But fall both on the same side of the Diameter*) Let the Diameter dce be drawn. Then will the whole Angle ace (at the Center) be double to the Angle ade (at the Circumference) by what was proved in the first Case.



Also the Angle bce is double to bde , by the same. Wherefore if from the Angle ace , we take away that bce , and from the Angle ade , which is the half of ace , we take away bde , which also is the half of bce , the remaining Angle adb must be just the half of acb . For 'tis as plain as an Axiom, that if one Quantity be double to another, and you take away from the bigger just the double of what you take from the other, the remainder of the Bigger must be double to the remainder of the Lesser.

3. If the Diameter fall between the Lines forming the Angle at the Circumference. Then will, as before, the Angle ace be double to abe (by Case 1 of this) and the Angle ecd will be double to ebd by the same; therefore the whole Angle acd must be double to abd . (So that in all Cases, the Angle at the Center is double to one at the Circumference if they both stand on the same Ark, or (which is all one) are in the same Segment.

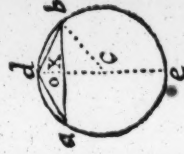


12. All Angles (in the same Segment or) Insisting on the same Ark ab are equal, let them Terminate in any part of the Circumference whatsoever.

For the Angle adb will be equal to aeb , because each is the half of the Angle at the Center acb (4. 11.)



13. An Angle at the Center bce standing on half of the Ark ab , is equal to the Angle adb at the Circumference standing on the whole Ark. For c is equal to twice x ; (by 4. 11) and x is equal to o , that is to half abd (4. 6, and 4. 8). Wherefore c is equal to adb . Q. E. D.



14. The Angle adb standing on the Semicircumference ae (or being in the Semicircle adb) is a right one. Let ce be drawn Bisecting the Semicircumference, then is (by the Precedent) the Angle ace at the Center, standing on half a Semicircle (or on a Quadrant) equal to adb at the Circumference, which stands on twice that Ark. But ace is a right Angle, wherefore adb , (its equal) must be so.

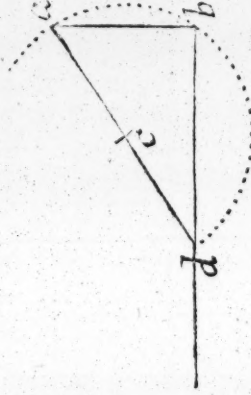


C O R O L.

COROLLARY I.

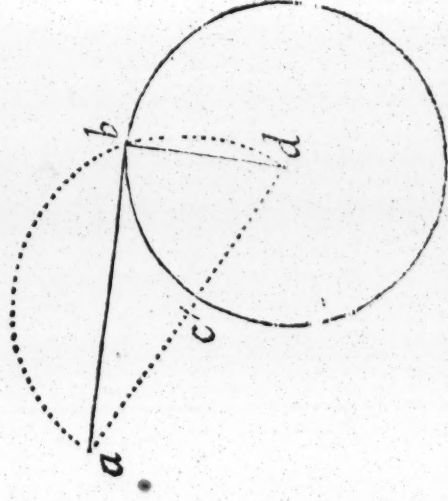
Hence is derived the Common Practice of Erecting

a Perpendicular, at a Point a , at the end of a given Line. For opening the Compasses to any convenient distance, set one Point in c , and with the other draw the Ark dba , cutting the given Line in d ; then a Ruler laid from d to c shall find the Point a , which is Perpendicularly over b : For the Angle dba , being in a Semicircle, is a right one.



COROLLARY II.

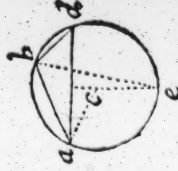
Hence also arises this expeditious Practice of draw-



ing from a Point given as a , a Tangent as ab , to a given Circle. For joining the Points a and d , the Centre of

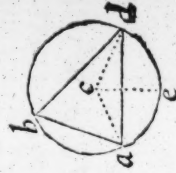
of the Circle, bisect their Distance ad in the Point c ; on c , as a Center describe the Semicircle abd : So shall ab be a true Tangent, because the Angle abd being in a Semicircle is a Right one.

15. The Angle abd in a Segment less (than a Semicircle) is *Obtuse*: Because the Ark aed being more than half the Circumference, its half, the Ark ae , must be more than $90d$, therefore the Angle abd , which is equal to aee , (4.13) must also be more than $90d$, that is *Obtuse*.



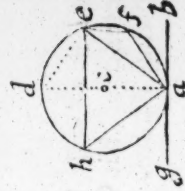
16. The Angle abd made in a Segment greater than a Semicircle, is *Acute*.

For 'tis equal to the Angle aee (4.13) whose Measure ae being the half of aed , an Ark less than a Semicircle, must be less than $90d$. And therefore abd is less than $90d$. (i.e.) *Acute*.



17. If a right Line as gb , touch a Circle, as in the Point a ; and another Line as ae cut it there. The Angle bae shall be equal to b , or any Angle made in the opposite Segment aec . And the Angle $ea g$ shall be equal to f , or any Angle made in the other Segment, efa .

For, drawing the Diameter ad , which will be Perpendicular to ab (4.9) (and also the Line de :) the Angle aed will be a right one; (4.14) and consequently, because the three Angles of every Triangle, are equal to right ones, (2.9) the Angle ead , together with d , must make just another right Angle.



But that Angle $d ae$ together with $ea b$ doth make also a right one, because the Radius ca is Perpendicular to the Tangent ab ; wherefore take away ead from both and then $ea b$ will remain equal to d . And consequently to b , or to any other Angle in that Segment.

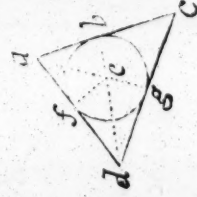
D

ment ahe , or that stands on the same Ark efa . For all those Angles are equal (by 4. 12). The Angle eab therefore, is equal to b : Which is the first part of the Proposition.

We must next prove the Angle gae to be equal to f ; which is the other part.

In the Triangle afe all the three Angles e, f and a , are equal to two right ones (2. 9). And the Angle e is equal to fab ; by the first part of this Proposition, for fa may be consider'd as cutting the Circle in the point a , where ab touches it, and consequently fab will be equal to any Angle that can be made in the opposite Segment $abdef$; and therefore to e . Now the two Angles efa and fab (that is e) together with f , are equal to two Right ones (2. 9), and so are efa and fad taken together with gae (1. 20). Wherefore the Angle f is equal to gae . Which was to be proved.

18. A Rectilineal Figure is said to



be *Circumscribed* about a Circle, when all its Sides touch the Circle without cutting it. Thus the Triangle dac is Circumscribed about the Circle bfg ; because every side of the Triangle touches the Circle, in b, g , and f .

19. A Figure is said to be *Inscribed* in a Circle when all its Angles are in the Circumference of that Circle, as the Triangle abc in the following Figure.



20. Every Triangle, abc may be Inscribed in a Circle; for if two Lines as eb and ei are drawn Perpendicularly Bisecting the sides ba and cb , they will cross or meet each other in the point e , on which as on a Centre, a Circle may be drawn, which shall pass through b . And

I say also, that That Circle shall pass through a and c .

For

For 1. The two Triangles eib and eia are equal; because ib is equal to ia by the supposition, the side ei is common to both, and the Angles at i are Right. Wherefore the side eb is also equal to ea (2. 11).

2. And for the same Reason *The Triangles ehc and ehb may be proved equal*, and consequently the side ec also will be equal to eb and to ea . But if those three Lines are all equal, the Point e , where they meet, must be the Center of a Circle of which they are *Radii*: And therefore the Triangle is Circumscribed by a Circle. Q. E. D. And thus may a Circle be made to pass through any three Points, if they be not all in a Right-Line.

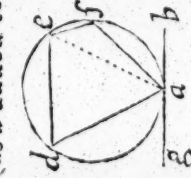
21. Every Triangle may (*have a Circle inscribed in it, or*) be Circumscribed about one. *vid. Fig. 18.*

For drawing the Lines ae and ed Bisecting the Angles a and d , and from the point e where they cross, letting fall the Perpendiculars (to the sides of the Triangle) eb , ef and eg : I say, that if you draw a Circle on the Center e thro' b ; that Circle shall touch all the sides of the Triangle in the points b , f , and g .

For 1. The two Triangles $ae f$ and $ae b$ are equal, as having the side ae common, the Angles at f and b right, and those at a equal (by the Supposition:) wherefore eb is equal to ef . (2. 14).

2. By the same Method, eg may be proved equal also to ef , (that is to eb) so that these three Lines being all equal, a Circle will pass thro' their three extremities, of which Circle they will be Radii, and being also all perpendicular to the sides of the Triangle, the said sides are Tangents to that Circle (4. 5), and therefore do *Circumscribe* it (by 4. 18).

22. Every Quadrilateral Figure $defa$ Incribed in a Circle hath its two Opposite Angles taken together (as d added to f) equal to two Right Ones.



For if thro' the Point a , there be drawn a Tangent as gb , and a Diameter as ea : The Angle at f will be equal to gae (4. 17) and the Angle gab will be equal to d (4. 17). And consequently the two Angles gae and gab being equal to two Right ones (I. 20), the Angles d and f taken together must be so too.

After the same manner might the other two opposite Angles daf and def be proved equal to two Right ones; By drawing another Tangent through the Point f .

23. The Converse of this Proposition is also manifest; viz. That if any Quadrilateral Figure have its Opposite Angles equal to two Right ones, it may the be inscribed in a Circle, That is, a Circle may be made that shall touch or pass thro' all its four Angular Points.

24. Every Polygon circumscribed about a Circle, equal to a Rectangle Triangle, one of whose Legs shall be the Radius of the Circle, and the other the Perimeter (or the Sum of all the sides) of the Polygon.



Let the Line FA be equal to the Radius fb , and it at Right Angles draw the Infinite Line $ABCD$, &c. out of which take Ab equal to ab , bB equal to bb , equal to bi and iC equal to ic , &c. So that the whole Line $ABCDEA$, may be equal to the whole Comp-

or *Perimeter* of the Polygon $abcdea$. Also draw FF Parallel to AA , So that all the Perpendiculars $FbFi$ Fk , &c. may be equal to the Radius fb or fi , &c. 'Tis then plain, that the Triangle AEB will be equal to the Triangle afb in the Polygon, and the Triangle BFC to bfc , and also CFD to efd , &c. So that all these Triangles taken together will be equal to all these in the Polygon, or to the whole Polygon.

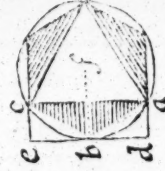
But the Triangle FAA is equal to all the five Triangles within the Parallels; because drawing the Lines, BF , CF , DF , &c. The Triangle FAB will be equal to FAB , FBC to FBC , &c. (3. 16); wherefore the Triangle FAA is equal to the Polygon, which was to be proved.

25. Every Regular Polygon is equal to a Rectangle Triangle, one of whose Legs is the Perimeter of the Polygon, and the other a Perpendicular drawn from the Center to one of the sides of the Polygon. The Proof of which is the same as that in the precedent Proposition; For all the Perpendiculars fb , fi , fk , &c. are equal, &c. See the *last Figure*.

26. Every Polygon Circumscribed about a Circle, is bigger than it; and every Polygon Inscribed, is less than the Circle. As is manifest, because the thing containing, is always greater than the thing contained.

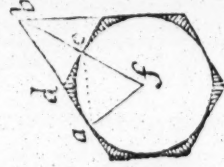
27. The *Perimeter*, or (as some call it, tho' improperly) the Circumference of every Polygon Circumscribed about a Circle, is greater than the Circumference of that Circle; and the Perimeter of every Polygon Inscribed, is less; as is plain from the 21st of the 2d Book.

28. If in any little Segment of a Circle, you Inscribe an Iſoſeles Triangle, as abc ; ſo that ab , be equal to bc : I ſay that Triangle ſhall be greater than half that Segment. For if you



draw a Tangent ebd , which ſhall be Parallel to ca ; and which ſhall be a ca is, Perpendicular to the Radius bf (4.5) (4.6) And then Complete the Rectangle $aedc$; That Rectangle will be greater than the whole Segment acb : But the Triangle abc , is the half of that Parallelogram (3.18). And therefore muſt be greater than half the Segment abc .

29. Let there be a Tangent adb , a Segment $fc b$, Chord ac , and another Tangent cd , I ſay, that the Triangle dbc is more than half the mixt Triangle acb , comprehended between the Lines ab , bc and the Ark of the Circle ac . For if the Triangle dbc , the Angle c , being a Right one (4.5), the ſide db , longer than dc (2.17). That is, that da ; which is equal to dc (4.7) Wherefore the Triangle dbc (*having a longer Baſe, but the ſame height with a dc*) muſt be greater than it (*as may be collected from* (3.17). And therefore it muſt be greater than the half of the whole Triangle acb . But the Triangle acb , is greater than the mixt Triangle, made by the Ark ac , and the Right Lines, ab and ac ; and therefore the Triangle dbc (which is more than half acb) muſt be greater than the half of the mixt Triangle abc . Q. E. D.



30. From theſe two laſt Propoſitions, it follows that by multiplying the ſides of Polygons, you may make them ſo *Circumſcribed* about, or *Inſcribed* in Circles that the difference by which the Circumſcribed exceeds or the Inſcribed wants of the Circle, ſhall be as ſmall as you will; Becauſe if from any Quantity whatever you take more than the half, and from the Remains

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more than its half, and again from that Remainder more than its half: You may by doing this very often at last come to leave a Remainder as small as you please; as is Self-evident. Thus (*See the 28th Figure*) After a Triangle is inscribed in a Circle that shall be less than it by the three great Segments, you may inscribe an Hexagon that shall exceed the Triangle by those three Segments, but shall be less than the Circle, by the six little Segments, that are left white in the Figure.

But those six white Segments taken together, do not contain so much space as the half of the three former Shaded ones, (4. 28). After this you may also inscribe a *Duodecagon*; which will be lesser than the Circle by 12 smaller Segments; which 12 Segments will still be less than the half of the six Segments of the Hexagon: And thus may you by increasing the Number of sides of the Polygon, lessen the Difference by which the Circumscribing Circle exceeds it, as much as you please. So likewise on the other hand, you might have first *Circumscribed* a Triangle, then an Hexagon, and then a Duodecagon, &c. (*and have made*, that way, *the Difference between the Circumscribing Polygon and the Circle, as small as you would*).

31. Every Circle is equal to a Rectangle Triangle, one of whose Legs is the Radius, and the other a Right-line equal to the Circumference of the Circle. For such a Triangle will be greater than any Polygon *Inscribed*, and less than any Polygon *Circumscribed*, (by 24, 25, 26 and 27 of this 4th Book) And therefore must be equal to the Circle.

For should it be greater than the Circle, be the excess as little as it will, a Polygon may be Circumscrib'd, whose difference from the Circle shall be yet *less* than the difference between that Circle and the Rectangl'd Triangle: And that Polygon will be less than the Triangle; which is Absurd. And if it be said that this Rectangl'd Triangle is less than the Circle; an Inscribed Polygon may be made, which shall be greater than that Triangle, which is impossible.

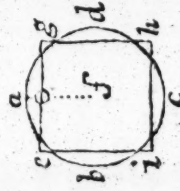
"This

" This kind of Demonstration which we here use, and which is call'd *Reductio ad Absurdum* *five ad Impossibile*, is one of the finest Inventions of the Ancients: And on it is founded all the *Geometry of Indivisibles*; so that I cannot but much wonder some of our Modern Authors should reject it as indirect and deficient. But if we must arrive to such a point of Niceness, that we can't bear any Demonstration, unless it be Direct and Positive; 'tis easie enough to give this before us such a turn, as shall render it Regular and Direct.

" For this cannot but be admitted as a Principle; *That if two determinate Quantities a and b are such, that every other imaginable Quantity, which is greater or less than a, is also greater or less than b; These two Quantities a and b must be equal.* And this Principle being granted, which is in a manner Self-evident, it may directly be prov'd that the Triangle (before mention'd) is equal to the Circle: Because every imaginable Inscribed Figure which is less than the Circle is also less than the Triangle: And every Circumscrib'd Figure greater than the Circle, is also greater than the Triangle.

This is that which is call'd the *Quadrature* of (or *Squaring*) the Circle, which consists in finding a Square Triangle, or any other Rectilineal Figure exactly equal to a Circle. And this would easily be done, could we find a Right-line equal to the Circumference; as is plain from this last Proposition. But such an Equality is not to be found *Geometrically*.

32. If a Right-line could be dispos'd into the Form of the Circumference of a Circle, it would contain more Space than any other Figure, or Regular Polygon whatsoever: Suppose the Circumference of the Circle $abcd$, to be dispos'd into the Form of a Square, or into any other regular Polygon: So that all the Sides eg, gb, bi and ie



together may be equal to the Circumference $abcd$.

I say

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I say the Circle is greater than that Square. For the Circle is equal to a Rectangle Triangle one of whose Legs is the Radius fa , and the other the Circumference. And the Polygon is equal also to such a Triangle, one of whose Legs is the same Circumference $abcd$, or the Summ of the Sides $geih$; and the other Leg is the Line fo (4. 25). But as the Line fo is less than the Radius fa , so the second Triangle, which is equal to the Polygon, must be less than the first which is equal to the Circle, and therefore the Square or Polygon must be less than the Circle, which was to be Demonstrated.

"And this is what we mean, when we usually say, that of *Isoperimetrical Figures* (or which have equal *Perimeters* or *Circumferences*) the greatest is the Circle.

BOOK

BOOK V.

Of Solids.

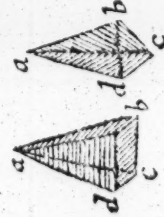
1. Right Line is said to be *Right* upon a

Plane, when it stands on it at Right Angles, just like a Pillar on the Ground, and is inclined no more to any one side of the Plane, than to the other.

2. Two Planes are Parallel to each other, when all the Perpendiculars that can be drawn between them are equal. (That is, *when they every where are equal distant.*)

3. One Plane is *Right* or Perpendicular to another Plane, when like a well-made Wall, it inclines and leans on one side no more than it doth on the other.

4. A *Solid Angle* is made by the meeting of three or more Planes, and those Joining in a Point; like the Point of a Diamond well cut.



5. If we imagine a Line as *ab*, fixt above in the Point *a*, to be moved along the Side of any Polygon *abc*; that Line by its Motion shall describe a Figure that is call'd a *Pyramid*.



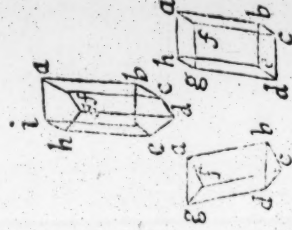
6. The *Polygon* is called the *Base* of the *Pyramid*.

7. If a Line fastned, as before, move round a Circle, as *abc*, it will describe a Cone; and the Circle

ELEMENTS, &c. 43

is its Base. And a Line drawn from the Center e to a , is call'd its *Axis*.

8. If a Line ab move uniformly about two Polygons gfa and $dc b$, which are every way equal, having their Sides and Angles mutually Parallel and Corresponding exactly to one another, as af to bc , fg to dc , &c. then that Line shall by its Motion describe a Figure which is call'd a *Prism*, and the Polygon is its Base.



9. If the Base of a Prism be a *Parallelogram*, then that *Prism* is call'd a *Parallelopiped*.

10. If a Line ab move uniformly round two equal and Parallel Circles, it shall describe or generate a *Cylinder*.



11. The Line joining the Centers e, c , in the two Bases, is call'd the *Axis*.

" There is no need of conceiving two Bases, Equal, Parallel and Opposite, for the Genesis of Prisms and Cylinders. For they will be describ'd as well by " Imagining a Line moving round the Circumference " of any plane Figure with a Motion always Parallel " to its self in its first Position. As if ab be supposed " to be carried round any of the Bases $dc b$, keeping " always the same Angle with the Plane, which it " first had, it will describe a *Triangular, Quinquangu-* " *lar, or Circular Prism*, according to the Figure of " the Base. And the upper End of the Line will de- " scribe a Base, (as you may call it) at the Top, equal " and Parallel to that below.

C O R O L

COROLLARY.

The Solid Content of all *Isoceles Prisms and Cylinders* (as also of all *Parallelopipeds*), is had by Multiplying their Height into the Area of their Base.

And if they are Scalenous Prisms or Cylinders, by Multiplying the Base by the Perpendicular Altitude.

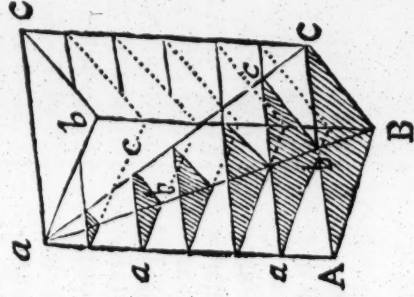
But after all, this Genesis of Prisms and Pyramids, respects only their Surfaces. And therefore the most proper way to conceive the Genesis of all Kinds of Prisms, is to imagine a Triangle, Quadrilateral Figure, or Polygon, or the Plane of a Circle to be moved in a Position always Parallel to it self; as suppose from b to e , or from g to d (in the preceding Figures) according to the Direction of the Line be or gd . Or according to *Euclide*, a Cylinder will be generated by the Revolution of the Parallelogram $gede$ (See Fig. in Art. 8.) round about the Axis ae .

And as for the Genesis of Pyramids, suppose the Triangle abc , to move downwards from the Top of

a Plane Angle, determined by the two Planes aAB , aBC : Let this Motion be always Parallel to it self, and let the Angular Point of the moving Triangle a be supposed always to keep in the Line aA .

'Tis plain, as this Triangle moves farther downwards, it will fill get more and more within the Solid Angle, and at last will come to be all of it within it, and to lie in the Position ABC .

which will be the Base of a Triangular Pyramid, whose Vertex is at a .



The

The same Triangle abc will also, by its Motion, describe another Pyramid, whose Base shall be the Parallelogram $bcbC$, and its Vertex a , as before.

13. If a Semicircle adb be turned quite round on its Diameter ab , it will describe a *Sphere* or *Globe* whose Axis will be ab , and its Center c the same with the Semicircle. Every Line passing thro' the Center c , and terminated at each end by the Surface of the Sphere, is call'd a Diameter, and may be call'd an Axis.



14. All Lines drawn from the Centre c to the Surface, are call'd Radii, and are all equal to one another.

15. Two Right-lines if they meet so as to cut or cross each other, are in the same Plain: Wherefore all the Angles and Sides of every Triangle are in the same Plane.

16. If two Planes ebd and agf cut or Intersect one another, they shall do so in a Right Line, as bd ; which is call'd their common Section.

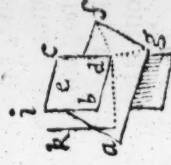
17. If a Right-line dc be Perpendicular to two Lines df and dg which are in the same Plane, that Line is also Perpendicular to that Plane.

18. If a Right-line dc be Perpendicular to three Right lines df , dg and da they are all three in the same Plane.

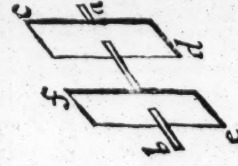
19. If two lines dc , bi are Perpendicular to the same Plane fga they will be Parallel to one another.

20. If two Lines dc , bi are Parallel; and you draw another Line, from any Point in one, to the other, as bd ; those three will be all in the same Plane.

21. If two Lines dc , bi are Parallel to a third ak , tho' that third Line be not in the same Plane with them yet they shall be Parallel to each other.



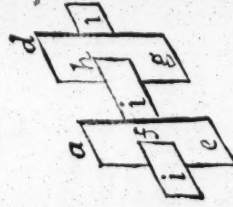
22. If a Line ab be Perpendicular to, (or make any other equal Angles with) two Planes fe and cd , those Planes are Parallel.



23. If two Parallel Planes, dbg and afe are cut by a third iii , the common Sections fe and bg are Parallel.

24. If a Solid Angle be made by three Plane Angles, any two of those are always greater than the third.

All these Propositions are so manifest to one that will but consider them with a little Attention, that 'tis needless to stay to Demonstrate them. (And indeed the Solemn and Regular Demonstration of a thing Plain in its self, always makes it more Obscure.)



25. The Plane Angles, concurring to make a Solid one, taken all together, are always less than four Right ones. For if they should make four Right Angles, they would form a Plane and not an Angle. Wherefore that they may make a Solid Angle they must be less than four Right ones.

'Tis a very good way in order to gain a clear Idea of Solids and their Angles, to make the Regular Bodies out of thick Paper or Past-board, as you are directed by Dr. Barrow, at the end of his thirteenth Book of Euclide; and by many other Authors.

26. In all Parallelopipeds the opposit Planes are equal; as is easie to conceive (from 5. 9).

The eight following Propositions (says our Author) are Demonstrated in the second Part of the Elements: And might be so here, by applying to Solids what is Demonstrated of Planes in the third and fourth Book. But there is no need to stay to do it now.

27. All Parallelopipeds having equal Bases (and Heights) or being between the same Parallels are equal (3. 14).

28. Every

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28. Every Parallelopiped is divided into two equal Triangular Prisms, by a Diagonal Plane, which is Perpendicular to its Base

29. Triangular Prisms having equal Bases (and Heights) or being between the same Parallels, are equal.

30. Pyramids having equal Bases and Heights are also equal.

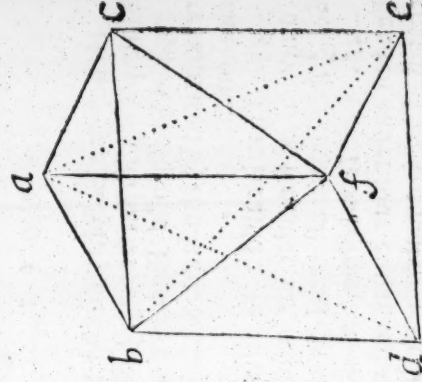
31. All Prisms in general, all Cylinders and Cones, with equal Bases and Heights, are equal.

32. Pyramids and Cones on equal Bases and of equal Heights with Prisms and Cylinders, are one third of such Prisms and Cylinders.

In a Triangular Prism, and Pyramid of the same Base and Altitude, it is thus prov'd.

The Quadrangular Pyramid $acfb$ is divided into two equal Triangular ones, by the Triangular plane fab , and the Pyramid $fcab$, is the very same with $bacf$;

and this is equal to the Pyramid $dfea$: as having an equal Base and the same Altitude with it, and therefore the whole Prism is divided into three equal Pyramids. And since all Multangular Prisms can be divided into Triangular ones, and that a Cylinder is only a Multangular Prism of Infinite Sides, the Proposition is Universally true; That Pyramids and Cones, &c.



C O R O L -

COROLLARY.

Hence the way of finding the Solidity of a Pyramid or Cone is discover'd, viz. to Multiply the Base by $\frac{1}{3}$ of the Perpendicular Altitude.

33. Every Sphere is equal to a Cone whose Perpendicular Axis is the Radius of the Sphere, and its Base Plane equal to all the Convex Surface of it.

For you may conceive the Sphere to Consist of an Infinite Number of Cones, whose Bases taken altogether compose the Surface, and whose Vertex's meet altogether in the Center of the Sphere: Just as a Circle may be imagined to be composed of an Infinite Number of Isosceles Triangle, the Aggregate of whose Bases makes the Circumference, and their common Vertex is at the Center.

COROLLARY.

Hence the Solidity of the Sphere will be gain'd by Multiplying its Surface by $\frac{1}{3}$ of its Radius.

34. Of all Solid Figures that can be encompass'd or terminated by the same Surface, the greatest is a Spherical one.

35. That is call'd a *Regular Body*, whose Surface is composed of Regular and equal Figures. And whose Solid Angles are all equal, as are —

36. The *Tetrahedron*, which is a Pyramid, comprehend'd under four equal and equilateral Triangles; so that its Base is equal to each Side.

Wherefore its Solidity will be found by Multiplying the Base by $\frac{1}{3}$ of the Altitude; which is the general way for all Pyramids.

37. The *Hexahedron* or *Cube*, whose Surface is compos'd of fix equal Squares, like Dice which are used in Play.

Its Solidity will be found by Cor. of Art. 12.

38. The *Octahedron*, which is bounded by eight equal and equilateral Triangles.

This Figure is two Pyramids put together at their Bases: Wherefore its Solidity is had by Multiplying the Quadrangular Base of either (here they are both join'd together in the Middle of the Figure) by one third of the Perpendicular Altitude of one of the joined Pyramids, and then doubling the Product.

39. The *Dodecahedron*, which is contain'd under twelve equal and equilateral Pentagons.

This Figure consists of twelve Pyramids, with Pentagonal Bases, whose common Vertex is the Center of a Circumscribing Sphere: Wherefore any one of these twelve pentagonal Bases, Multiplied by $\frac{1}{3}$ of the distance between the Center of that Base, and the Center of the Sphere; and then that Product Multiplied by twelve, gives the Solid Content of this Regular Body.

40. The *Icosahedron*, consisting of twenty equal and equilateral Triangles.

This Figure is composed of 20 Triangular Pyramids, all equal to one another, and whose Vertex is the Center of a Circumscribing Sphere: Wherefore any one of the 20 Triangular Faces, Multiplied by $\frac{1}{3}$ of the distance between the Center of the Face and the Center of the Sphere, and that Product multiplied again by 20, gives its Solid Content.

41. Besides these five Regular Bodies, 'tis not possible to find any others that shall correspond to the Definition; which is thus Demonstrated.

To begin with equilateral Triangles, which are the most simple of all Rectilineal Figures. Of these there must be three at the least to make a solid Angle, and three of them join'd together will just make the *Tetrahedron*. For those three Triangles meeting in a Point to form a Triangular Base similar and equal to the sides; as appears by the bare composition of the Figure: four such Triangles join'd together in a Point make the Angle of the *Octahedron*.

By joining five such Triangles together the Angle of the *Icosahedron* is form'd.

E

But

But fix such Triangles join'd in a point cannot make a Solid Angle: Because they make four Right ones (*for every Angle of an equilateral Triangle is $\frac{1}{3}$ of two, or $\frac{2}{3}$ of one Right Angle, either of which fractions Multiplied by six gives four Right Angles.*) Whereas every Solid Angle is made up of such plane Angles as all together must be less than four Right ones (5. 25). So that with Triangles 'tis Impossible to form any more Regular Bodies than these three.

Next if you take *Squares* and join three of them together, they will make the Angle of the Cube: And there can no other regular Body but a Cube be made with Squares, for four Squares join'd together will not make a Solid Angle but a Plane. (5. 25).

If you join the Angles of three Pentagons together, you will constitute the Angle of the Dodecahedron: But four such Angles cannot make a Solid one.

And lastly, *Three Hexagons* joined together do make just four Right Angles, and therefore they cannot make a Solid Angle: And as for three *Heptagons*, or other Figures of yet more Sides, they can much less do it; (*because their Angles being very obtuse, three of them will exceed four Right ones.*) So that upon the whole 'tis plain, that of these five *Regular Bodies*, three are made of Triangles, one of Squares, and one of Pentagons, and *there can be no other.*

BOOK VI.

Of Proportion.

WHEN we speak of *Magnitude*; and say that any Quantity is *great*, we always make a Comparison between that Quantity and some other of the same Nature, in respect to which we say that it is *great*.

Thus we say of an *Hill*, that 'tis *Little*; or of a *Diamond*, that 'tis *Large*; because we compare that *Hill*, with others that are *Higher*; and in respect of them 'tis *Little*; and we compare that *Diamond*, with others that are *Little*, and in respect of them, we say it is a *Large* one.

2. When we consider one Quantity in respect of another, to see what Magnitude it hath in comparison of that other: That Magnitude so found, is call'd its *Ratio*, or *Reason*; tho' it would be more intelligible if it were call'd *Comparison*.

3. That Quantity which is compar'd with another is call'd the *Antecedent*; and *that other* with which it is compar'd, is call'd the *Consequent*.

4. When we consider four Quantities, and compare them (*by Pairs*) two with two; as *a* four, with *b* 2; and *c* 6, with *d* 3. we find that *a* hath as much Magnitude (or is as big) in comparison of *c* as *b* hath in comparison of *d*: Then we say that their *Ratios* are Equal; that is, the *Ratio* that *a* hath to *b*, is equal to the *Ratio* that *c* hath to *d*.

$$\begin{array}{ccccccc} 6 & & 3 & & 6 \\ \text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} \\ 4 & & 2 & & 1 & & 1 \\ \text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} & \text{I} \\ & & a & b & c & d & e \end{array}$$

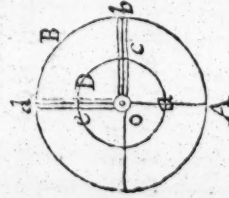
of c to d : For as a is twice as big as b ; so c is twice as big as d .

5. But if a 4, hath more Magnitude in respect of b 2, than c 6 hath in respect of e 5. That is, if as a 4 is twice as big as b 2, c 6 be found not to be twice as big as e 5. Then the *Ratio's* are unequal: And we say a hath a greater *Ratio* to b , than c hath to e . So that to have a greater *Ratio*, is nothing but to have more Magnitude, or to be bigger, in respect of a second Term, than a third is in respect of a fourth.

6. The Equality of *Ratio's* is call'd *Proportion*; and when we find that of four Quantities or Numbers, the first hath as much Magnitude (*or is as big*) in respect of the second; as the third is in respect of the fourth; then we say that those four Quantities are *Proportionals*.

The better to make the *Mysteries of Proportion comprehend'd, which pass for the most difficult things in Geometry, as unquestionably they are most Important, I will explain them by an Example; which (in my Opinion) will render all those things very Intelligible, which otherwise appear very perplex'd.*

7. Let us imagine the Circle $b A d$ to be describ'd by the Motion of the Line $o b$, round the Center o . And at the same time, let the Circle $c a c$ be describ'd by the Motion of a Point c , in the Line $o b$. Let us suppose also that the Line $o b$ be mov'd once round again, and at last to stand in the Position $o d$. Let the Ark $d B$ be call'd B , and the Ark $e D$, be call'd D . Let A be put for the whole outer Circle, and a for the whole inner one.



Now if we compare the whole Circle A with its Ark B , and the whole other Circle a with its Ark D . We shall find plainly, that the Circle A is just as big in respect of the Ark B , as the inner one a is in respect of the Ark D ; and therefore if B be a fourth, or any other part

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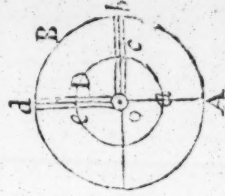
part of the Circle A, D also will be a fourth, or the same proportionable part of its Circle *a*. Which we usually exprels by saying, as A is to B, so is *a* to D. And write it thus A. B :: *a*. D.

8. If you should change the Order of the Terms, and compare B with A and D with *a*; you will find plainly that B. A :: D. *a*. So that supposing A. B :: *a*. D. we cannot but presently conclude by *Inverse* Proportion, that B. A :: D. *a*.

9. If you change them so as to compare Antecedent with Antecedent, and Consequent with Consequent; you will find *Alternately*, that A. *a* :: B. D. And this is very plain; for if the whole Circle A be double, triple of, or in any other Proportion, to the Circle *a*, the Ark B must be also double, triple of, or in the same Proportion to the Ark D. This I say is plain, because the two Circles A and *a* are describ'd by the Motion of the Line *ocb*; so that while *b* describes the Circle A, *c* describes the inner Circle *a*; and while *b* describes the Ark B; *c* also describes the Ark D. And this by one common circular Motion; only the Point *c* moving much slower than the Point *b*, describes a Circle much less, in proportion to the slowness of its Motion: Thus also when the Point *b* shall have describ'd the Ark B, the Point *c* in like manner will have describ'd the Ark D, which will be much less than B; in proportion to the slowness of its Motion.

10. If we compare the differences between the Antecedents and Consequents, with their Consequents; as for Instance, A less B with B, and *a* less D with D, we shall find they also are Proportional: And that A less B. B :: *a* less D. D.

For 'tis manifest that the Ark *bA* d (which is A less B) is to B as the Ark *cae* (which is *a* less D) is to D. And this is call'd *Proportion by Division*.



11. If we add the Antecedents and Consequents together; we shall find that A, more P. is to B :: as *a*, more D is to D. Which is call'd *Composition*.

12. And if we should say, that A. A less B :: *a*. *a* less D. This kind of Proportion is call'd *Conversion*.

13. If never so many Quantities, are thus Proportional: It will be as any one Antecedent to its Consequent :: So is the Summ of all the Antecedents to the Summ of all the Consequents. *v. gr.*

If 4. 12 :: 2. 6 :: 3. 9 :: 5. 15 :: then shall
14. 42 :: 4. 12.

14. If *a. b :: c. d*

4. 12 :: 3. 9 and also

b. f :: d. g

12. 36 :: 9. 27

Then will it be by Proportion of Equality.

a. f :: c. g.

4. 36 :: 3. 27.

The Reason of which is plain, if you consider, That since *b. f :: d. g : f*, and *g*, must needs be either *Similar* or *Aliquot Parts*, or *Equimultiples* of *b* and *d*. And therefore since *a* and *c*, are to *b* and *d*, in the same Ratio, *b* and *d* are to *f* and *g*, *a* must alio be in the same Ratio to *f*. the *Part* or *Multiple* of *b ::* as *c* is to *g*. the *Part* or *Multiple* of *d*.

15. If B be taken as often as D. *ex. gr.* 3 B and 3 D we may conclude that B. D :: 3 B. 3 D. or as 103 to 103 also as 12 $\frac{1}{2}$ B. to 12 $\frac{1}{2}$ D. And so on in whatsoever Proportion the two Magnitudes B and D are multiplied, so they are multiplied equally, or that you take one as often as you take the other. For then there will be the same Proportion between the Magnitudes thus equal

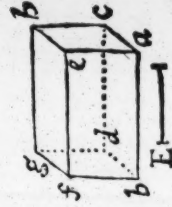
multipled, as there was between the simple Magnitudes, before such multiplication. And these Magnitudes thus equally multiplied are call'd *Equimultiples* of the simple Magnitudes B and D; and we say that *Equimultiples* are in the same Proportion as such simple Magnitudes, out of *which they are compounded*.

16. If B be divided in the same manner as D is; and ex. gr. you take a fourth part of B and the like of D. or the tenth or any other part of B and the same of D. Then will these Parts be Proportional to their wholes, B. D :: $\frac{1}{4}$ B (or $\frac{1}{10}$ B.) is to $\frac{1}{4}$ D. or $\frac{1}{10}$ D. All which is Self-evident.

17. To multiply one Line by another is to make a Rectangl'd Parallelogram, whose two Contiguous Sides shall be the two Lines given. Thus, if you multiply the Line A by B; 'tis the same thing as to make the Rectangle *abcd*; whose Side *ab* is equal to A, and *ac* to B.

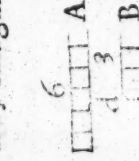


18. To multiply a Rectangle, or any other Surface by a Right line, is to make a Rect-angled Paralleliped (or Prism) whose Base shall be the Surface given, and its Perpendicular height the Line given.

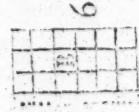


Thus to multiply the Surface *abcd* by the Line E, is the same thing as to make a Solid *abfghe* whose Base is the Surface given *ad*, and its height *ae* or *bf*, equal to E the Line given.

19. And the better to conceive these Multiplications we may imagine the two Lines as if they had some



little breadth, and their whole lengths to be divided into little Squares, as you see in these Figures: Where A is a Line (or rather a Ruler) containing six small Squares; and B is another Ruler containing three small Squares of the same breadth with those in A.



Wherefore to multiply B by A, or A by B, is to take the Rule B, as often as there are little Squares in A, or which is all one at last, to take A as often as there are Squares in B; so B, being taken six times makes



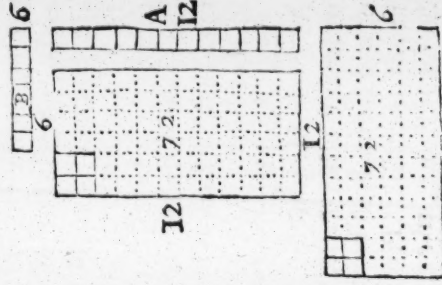
the first Rectangle, containing 18 such little Squares; and A taken three times will make another Rectangle, containing also 18 Squares, and in the whole Equal to the former.

20. Observe here, that the same Multiplication will be made, tho' the length of either Line doth not consist exactly of such a certain Number of small Squares. As for instance, suppose in A there be just Six Squares, but in B three and an half, a Quarter, or any other Part: Or any Excess, be it what it will, and there mark'd with *d* (see *Fig.* preced.)

Then need B be taken only Six times (as before) to be multiplied by A, and the Product will be the first Rectangle B, containing 18 whole Squares and six of the Parts or Excesses *d*. So in like manner if A had been multiplied by B, that is taken 3 times and a half, or 3 times, and the excess *d*; The Rectangle A would be Produced, consisting as the former of 18 whole Squares, and of six Excesses of *d*. That is, *each Rectangle will contain in the whole (if d be half of one of the Squares) just 3 whole Squares, above the former Area.*

21. If we imagine the little Squares in A to be subdivided into others, that are just half of the former; after this, its length remaining still the same, it will have 12 small Squares, and if B have its Divisions or Squares lessen'd in the same Proportion: Then in B there will be six small Squares: So that now, if A be multiplied by B, or B by A; Two Rectangles will be produced equal in the whole to the former, for A taken six times, makes the first Rectangle containing 72 small Squares; and B taken 12 times, makes the other Rectangle containing also 72 Squares: Now these 72 Squares are in reality neither more nor less than the 18, of the other Rectangles, (tho' drawn larger that the Subdivisions may appear) because one of those 18 contain'd as much Space as, or answer'd to 4 of these 72, as is apparent from the Black Lines in the Figure: So that whatsoever breath, tho' never so small, be given to these Lines, subdividing the little Squares infinitely, 'tis plain the Rectangles made out of them by Multiplication, will still be the same. And from hence we may boldly suppose the Lines, for Indivisibles; and that they may be multiplied one into another, when a Rectangle is made out of them; because the Magnitude of this Rectangle never varies, tho' some very small Breadth be allow'd to the Lines.

22. 'Tis very easy to apply all this to the Multiplication of Solids; But then instead of Squares, we must Imagine Cubes: For if you conceive a Surface compos'd of 12 Cubes, and on the other side a Line consisting of two Cubes: The Superficies consisting of 12 Cubes will be multiplied by the Line of two, by taking that Super-



Superfices as often as there are little Cubes in the Line; *i. e.* Twice, and so a Solid will be produced consisting of 24 small Cubes.

23. By all which it appears, that these small Squares and Cubes in the multiplication of Lines and Surfaces, are the same things as Unites in the Multiplication of Numbers. For as to multiply one Number by another *v. gr.* 3 by 5, is only to take 3 as often as there are Unites in 5, or to take 5 as often as there are Unites in 3; and either way the Product will be 15: So to multiply one Line by another is to take either of those Lines, as often as there are little Squares in the other: And to multiply a Surface by a Line, is to take that Surface, as often as there are (*supposed*) Cubes in that Line.

24. All Magnitudes may be express'd by Lines; As if one Magnitude be double or triple of another, or in any other *Ratio*, two Lines may easily be taken, of which one shall be double or triple of the other, or in any other like Proportion with those Magnitudes: So for Instance, to express two times, as one Hour and two Hours; or two Velocities, of which one shall be double to the other; you need only take two Lines, as *a* double of *b*; and then you may say that *a* represents two Hours or Velocities, and *b* answers to one of each: and then you may proceed to compute with those two Lines, as with the Hours and Velocities themselves, &c.

25. To know the Proportion of Rectangles, the *Ratio* of the Length of One to the Length of the Other, and moreover the *Ratio* of the Breadth of One to the Breadth of the Other must be known.

For Example; To know what Proportion the Rect-



angle *ac* hath to *eg*: 'Tis not enough only to know that the Length *ab* is triple of *eh*; but it must be known also, that *ad* is double of *ef*. For if *ai* be taken equal to *ef*, the Rectangle *bi* will be triple of *eg*, because *ab* is triple of *eh*, and *ai* equal to *ef*. And moreover be-
cause

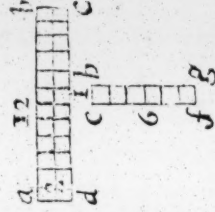
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cause id is also equal to ai , or ef (for ad is supposed to be double of ai , and of ef) the Rectangle ic shall also be triple of eg ; so that the whole Rectangle ac is twice triple of the Rectangle eg ; that is, sextuple of it, or containing it six times. And what we say now only of the double or triple *Ratio* of their Breadths and Lengths, is also to be understood of any other *Ratio*, be it what it will: For if ab be quadruple of eb , and ad triple of ef , the Rectangle ac , will be three times quadruple of the Rectangle eg : that is duodecuple of it, or doth contain it twelve times.

But if ab be duodecuple of eb , and at the same time ef be Triple of ad , then there is a certain Compensation made: For if Respect were had to their Breadths ab and eb only, the Rectangle ac would exceed the other, nay indeed contain it 12 times: Nevertheless this excess is lost (*in some Measure*) in respect of their Altitudes or Heights ad and ef , which if only consider'd, the Rectangle eg would be Triple of ac . But then when we come to compare these several Excesses and Deficiencies together; we shall find that the Rectangle ac being one way 12 times greater, and the other way three times less than eg , will be at last but only four times as great.

26. And this is what we mean, when we say, that all Rectangles are to each other in a *Ratio Compounded* of that of their sides; for if ab be triple of eb , and ad double of ef , the Rectangle ac , shall be to the Rectangle eg in a *Ratio* compounded of the triple and the double, that is, it shall be thrice double, or twice triple, or in one Word Sextuple.

So also if ab were quadruple of eb , and ad triple of ef ; the Rectangle ac would then be to eg in a *Ratio* compounded of the Quadruple and the Triple; so that



that it would have been three times Quadruple, or four times Triple, or in one word Duodeuple of eg .

Moreover, if ab were Duodeuple of eh , and ad Subtriple of ef , (that is, if ef be Triple of ad) the Ratio of the Rectangle ac to eg would be compounded of the duodeuple and subtriple Ratio; so that ac , would have been 12 times subtriple of, or in one word Quadruple of eg .

If you take the third part of a Crown 12 times it will make, or be equal to four whole Crowns: So that four Crowns are 12 times subtriple of one Crown; that is, do make 12 thirds of a Crown.

27. From whence it will appear that if the sides of two Rectangles are reciprocally Proportional, those two Rectangles are equal: For if ab be double to eh ,

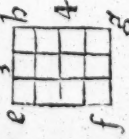
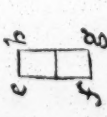
and reciprocally hg be double to cb :

Or if ab be triple of eh , and then hg be triple of bc ; or in a word, if whatever Ratio ab hath to eh , hg hath back again the same Ratio to bc , 'tis plain that as much as the first Rectangle ac exceeds the other in length, just so much is it ex-

ceeded by the other in breadth; so that the length of one Compensates for the breadth of the other, and consequently it must be equal. And from hence is deduced this most useful and important Proposition: That,

28. If four Quantities (or Numbers) be proportional, the Product arising from the Multiplication of the two middle Terms, is always equal to that, which is made by the Multiplication of the two Extreams. As if $ab.ec :: hg.bc$.

I say, from the Multiplication of the extreams ab by bc there is produced the Rectangle ac : And by multiplying the middle Terms eh and hg , there is produced the Rectangle eg ; and those two Rectangles ac and eg are equal.



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equal. (6. 27). (Because, as much longer as a b is than e h, just so much longer is h g than b c.)

C O R O L L A R Y.

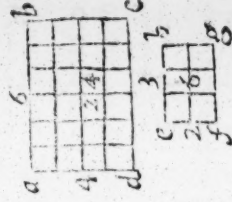
Hence, if in two ranks of *Discrete* Proportionals, the four middle Terms are the same: As if $a. b :: c. d$, and then also $e. b :: c. f$. I say it will be as $a. e :: d. f$. So will reciprocally $f. b$ to d : For since the middle Terms are the same in both, the Rectangle $a d$ will be equal to $e f$, and consequently their Sides must be reciprocally proportional; that is $a. e :: f. d$.

What is thus done by Lines and Rectangles, may be done by any Quantity whatsoever; because all Quantities can be express'd by Lines, and all Multiplications of Magnitudes by Multiplications of Lines, i. e. by Rectangles. (6. 24).

29. When Rectangles have their sides Proportional, so that $a b. e b :: a d. e f$. then is the Rectangle $a c$ to the Rectangle $e g$, in a *Duplicate Ratio*, to that of their Sides: For the Ratio of $a c$ to $e g$, is Compounded of the Ratio of $a b$ to $e b$, and of the Ratio of $a d$ to $e f$.

(6. 26). But the Ratio of $a b$ to $e b$ is in this Case, (by the Supposition) the same as the Ratio of $a d$ to $e f$; so that to gain the Ratio which the Rectangle $a c$ hath to $e g$, We need only take *twice* the Ratio of $a b$ to $e b$. For Example, if as here $a b$ be double to $e b$, and $a d$ double to $e f$, the Rectangle $a c$ shall be twice

double, that is, Quadruple of the Rectangle $e g$. And if $a b$ had been triple of $e b$, and consequently $a d$ triple of $e f$: Then the Rectangle $a c$ would have been three times triple, that is nine times as big as $e g$: Or if $a b$ had been Quadruple of $e b$, $a c$ would have been 16 times as great as $e g$.



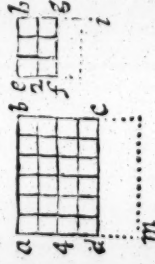
30. If

30. If a third Line be taken as no ; and it be so proportional that $ab, eb :: eh, no$. Then shall the two Rectangles ac and eg be to one another as the two Lines ab and no .
(*vid. Fig. Preced.*)

For a is to n in a duplicate Ratio of a to e . And if ab had been, (as it is double,) triple or quadruple of e : Then would ab have been in a Ratio three times triple; or four times quadruple of (that is nine or 16 times as great as) the third Proportional n .

31. Those Rectangles which have their sides thus Proportional: That $a b . e b :: a d . e f$. are call'd *Similar*, whose *Homologous Sides* are those which answer each to other in the Proportion, as $a b$ and $e b$, or $a d$ and $e f$: For as $a b$ is the greatest side of the Rectangle $a c$, so $e b$ is also the greatest side of the Rectangle $e g$.

32. All Squares are similar Rectangles. For 'tis plain that if ab be double or triple of eb , am must also be double or triple of bi : Because am is equal to ab , and bi to eb .



33. All similar Rectangles are to each other as the Squares of their Homologous Sides. I say the Rectangle ac is to the

Rectangle eg :: as the Square bm to the Square ei .
 For as well Squares as Rectangles are to one another
 in a Duplicate Ratio of ab to eb (6.29, 30).

34. To know the *Ratio* between two solid Rectangles, or *Parallelopipeds* there ought to be known the several *Ratio's* that their Bases and Heights have to each other; Because the *Ratio* of one Solid to another is compounded of the *Ratio's* of their Lengths and Breadths and Thicknesses or Heights; as is easie to conceive, if that be well understood which hath been laid about the Proportions of Rectangles. For if one Parallelopiped hath its Base double to the Base of another, and its Height, triple of the Height of the other; The former will be

be twice triple, or three times double, or in one word Sextuple of the latter.

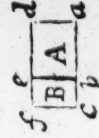
35. If the *Bases* of two Parallelopipeds be *Reciprocally* as their *Heights*, those Parallelopipeds are equal; Which is prov'd by the 27th of this Book; for as much as one exceeds the other in Breadth and Length, so much doth the other exceed it in height.

36. When Parallelopipeds have all their sides Proportional, they are call'd *Similar*; and they are in a *Triplicate Ratio* of their Sides, as it hath been prov'd of Rectangles, that they are in a *Duplicate Ratio* of their sides.

37. Similar Parallelopipeds are to one another as the Cubes of their Homologous Sides; for both Cubes and Parallelopipeds are in a *Triplicate Ratio* of their Homologous Sides.

38. All Rectangles having the same Heights, are to one another as their Bases.

Let the Rectangles A and B be between the same Parallel Lines *df* and *ca*; so that *ad* be equal to *cf*; then do I say, that $A.B :: ab.bc$. That the Rectangle A. is to the Rectangle B. as the Base *ab*. to the Base *bc*: And that if, for Instance, *ab* be double to *bc*, then shall A be double to B. For A is nothing but the Line *ba* multiplied by *da*. (6. 17); and B is nothing but the Line *cb* multiplied by the same Line *ad* or (which is all one) *be* or *fc*. Wherefore (6. 15) $A.B :: ab.bc$.



39. All *Parallelograms* which are between the same Parallels (or which have the same Height) are as their Bases. I say the Parallelogram *eb* is to the Parallelogram *bg* :: as the Base *ab* is to the Base *bc*. For having made the two Perick'd Rectangles on the same Bases, those will be equal



to the Parallelograms, (by 3. 14). But those Rectangles are as their Bases, (by the Precedent). Wherefore the Parallelograms must also be as their Bases; That is $eb.bg :: ab.bc$.

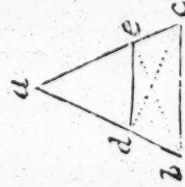
40. All

40. All Triangles which (*have the same Heights*) or are between the same Parallels, are as their Bases; For they are halves of Parallelograms. (3. 8).

41. When Triangles (as those in the following Figure) have their Bases on one and the same Line, and their *Vertices* or tops meeting in the same Point; they are then taken to be between the same Parallels, as ade and cde , and ade and bde (*because they have the same Perpendicular Height.*)

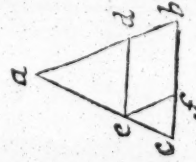
42. If in any Triangle a Line be drawn Parallel to the Base, that Line shall cut the Legs Proportionally.

Let the Triangle be abc , and let the Line de be Parallel to bc . I say that $ad : ae :: ab : ac :: db : ec$, &c. Draw the Lines dc and eb , then shall the Triangle ecd be to ead , as the Base ec , is to ae . (6. 40, 41). So also will the Triangle deb be to dea : as the Base db , is to da . But the Triangle ecd is equal deb (3. 16); wherefore the Triangle bde (or ced) is to the Triangle ead : as bd , is to da : or as ce to ea . Therefore also must $bd : da :: ce : ea$, because both the *Ratio* of bd to da , and also that of $ce : ea$, are the very same with that of the Triangle bde or ced , to the Triangle ade .



Triangle bde (or ced) is to the Triangle ead : as bd , is to da : or as ce to ea . Therefore also must $bd : da :: ce : ea$, because both the *Ratio* of bd to da , and also that of $ce : ea$, are the very same with that of the Triangle bde or ced , to the Triangle ade .

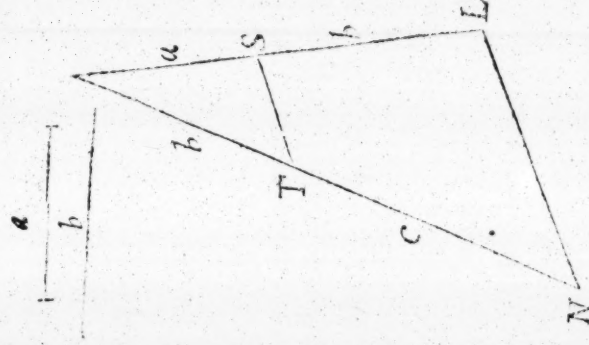
43. If in a Triangle, as acb , you draw a Line de Parallel to the Base cb , I say, that $ed : cb :: ae : ac :: or as ad : ab$. For drawing ef Parallel to ab ; fb will be equal to ed (3. 9). But by the Precedent $fb : cb :: ae : ac$, wherefore $ed : (fb) : cb :: ae : ac$, or as ad , to ab .



PROBLEM I.

Two Lines a and b being given, to find c, a third Proportional to them.

Make any Rectilineal Angle, and from the Vertex or Top of it, set the two given Lines down on the Legs, as you see in the Figure. Set also *b* downward

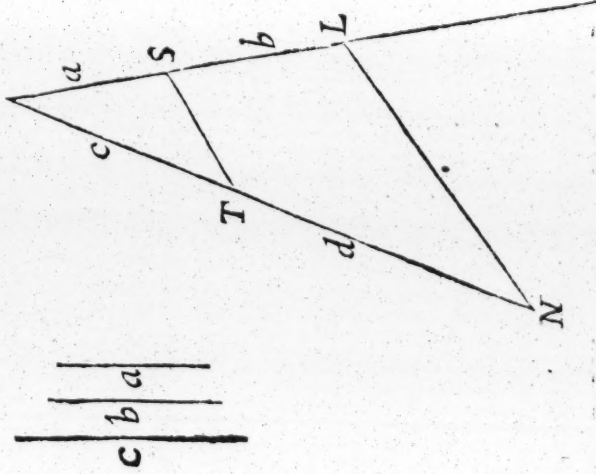


from *S* to *L*, join *ST*, and draw *NL* Parallel to it, so shall *TN* be *c*, the Line sought : For $a.b :: b.c$. by this Proposition.

PROBLEM II.

If three Lines as a and c had been given, to find a fourth Proportional (as d) to them, or to work the Rule of three Lines, you must proceed thus.

Set the two first Lines a and b from the Vertex down on the same Leg; and then set c the third Line, from the Vertex on the other Leg: Draw the Line TS , and



thro' the Point L draw LN Parallel to it; So shall TN be equal to d the fourth Proportional sought; for: $a.b :: c.d.$ by the Precedent Propositions.

PROB

PROBLEM III.

And this way 'tis very easie to find a Line that shall express the Product of any two Numbers or Quantities: Or the Quotient of one Divided by the other.

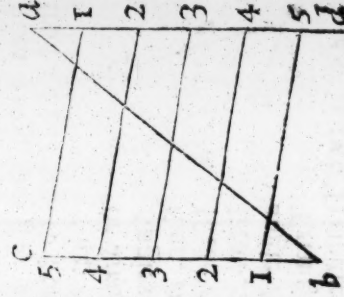
For since in all Multiplication; as 1. is to the Multiplier :: So is the Multiplicand to the Product: And since in Division, as the Divisor is to 1. :: So is the Dividend to the Quotient: You may take your 1. of any Length off a Scale, and finding a fourth Proportional to the three first Terms, that shall be a *Product*, or a *Quotient* required; Thus if *b* were to be multiplied by *c*: make *a*, equal to Unity, and set off *b* and *c* as before shew'd, so shall *d* be the Product. Or if *d* were to be divided by *b*; take *a* = 1, and set off all things as before; so shall *c* be the Quotient; for *b.a :: d.c*.

PROBLEM IV.

To divide a given Line a b into any Number of equal Parts: As Suppose into Six.

Make at *a* and *b* any two equal Angles, and on the

Legs *da* and *eb* run with a pair of Compasses, five equal Divisions (for they must be always one less in Number than the required Division or Parts of the Line given) drawing also Lines across from one Point to the other, as you see in the Figure; so shall those Lines divide the given Line *ab* into the six Parts required: For the crossing Lines being Parallel one to another,



must

must divide ab in the same Proportion as ad and be are divided.

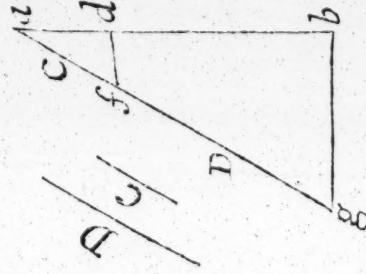
PROBLEM V.

To Divide a given Line ab into two Parts, so that they shall be to each other as the Line C to D ; or in any given Ratio.

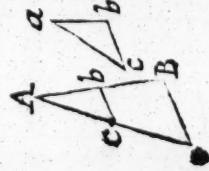
Make any Angle with the given Lines ab , and let the Line c from its Vertex a to f : And let the Line D from f to g ; draw the Line gb , and thro' f , a Parallel to it, as fd : So shall the Point d divide ab in the Ratio required: For $C, D :: ad, db$.

And much the same way may you cut off from any given Line ab any part or parts required; as suppose $\frac{2}{3}$.

Make an Angle gab as before, and let on the Leg ag , ag equal to five parts taken off from any Scale: Then set two such parts from a to f , join gb , and draw fd Parallel to it; so shall ad be equal to $\frac{2}{3}$ of ab .



44. Those Triangles are call'd Like, or Similar, which have all their three Angles respectively equal one to another, or which are Equiangular: v.g. If the Angle A be equal to a , the Angle B to b , and C to c , then the whole Triangle ABC is Like or Similar to the Triangle abc .



45. If two Triangles have two Angles in one equal to two in the other, the third Angle in one must be equal to the third in the other; and the Triangles will

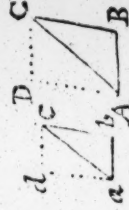
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be Similar : For Since the Sum of all the three Angles in each, is equal to two Right ones. (2. 9). If two Angles in one, be equal to two in the other, the remaining third Angles will be equal.

46. All similar Triangles have their Sides about the equal Angles Proportional. I say, $AB. ab :: AC. ac :: BC. bc$, &c. For take, in the greater Triangle ABC , Ab equal to ab , and Ac equal to ac ; then will the Triangle Abc be every way equal to abc (2. 11). and the Angle Abc is equal to the Angle abc : Wherefore it will be alio to B , which by the supposition was equal to ab , and therefore cb is Parallel to CB (1. 31). and Consequently (by 6. 42, 43), $Ab. AB :: Ac. AC :: cb. CB$.

47. All similar Triangles are in a duplicate Ratio of, or as the Squares of their Homologous Sides.

Let abc be similar to ABC ; so that $ab. AB :: bc. BC$. Then first, if the Angles B and b are Right ones, compleat the Rectangles db and DB ; which Rectangles will be to one another in a duplicate Ratio of the side bc to the Homologous Side BC , or as bc Square to BC Square (6. 29, 31). But the Triangle abc is just half the Rectangle bd , and the Triangle ABC is just half the Rectangle DB (3. 8); wherefore these Triangles must also be to each other in a duplicate Ratio of, or as the Squares of their Homologous Sides.

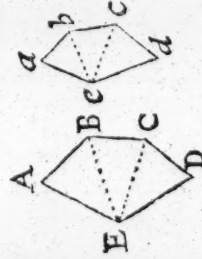


2. If the Triangles are not Rectangular, as in the Figures adjoined, draw the Parallels ad and AD , and compleat the Rectangles $bedc$ and $BEDC$; then, the Triangles adc and ADC will be Similar; because the Angle d is equal to D (as being both Right ones) and also the Angle

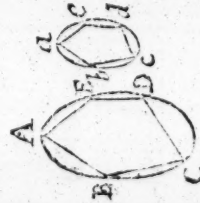


ca is equal to DA C, because they are severally equal to the (*Suppos'd*) equal Angles acb and ACB (1. 31). wherefore ac . $AC :: cd$. CD (6. 46). But ac . $AC :: bc$. BC (by the Supposition) and therefore cd . $CD :: bc$. BC . And consequently the Rectangles bd and BD are Similar (6. 31). and therefore are as the Squares of their Homologous Sides (6. 33). And therefore their halves must be so too: *i. e.* the Triangles $a^b c$ and $A B C$ (3. 18). are to each other in a Duplicate *Ratio* of, or as the Squares of their Homologous Sides. Q. E. D.

48. Similar Polygons are those which having an equal Number of Sides have all the several Angles in one, equal to those in the other, and also the sides about those equal Angles Proportional. As if the Angle A be equal to a , B to b ; and moreover AB . $ab :: BC$. $bc :: CD$. cd . then those two Polygons are Similar.



49. And among *Curvilinear* or *Mixt Figures*, those are *Similar* in which you may Inscribe, or about which you may Circumscribe similar Polygons; so that any Polygon being Inscrib'd or Circumscribed about one Figure, you may Inscribe or Circumscribe a Similar one about the other. For instance, if having Inscribed any Polygon as $ABCDE$ in the greater *Curvilinear* Figure, you can Inscribe another in all respects Similar to it in the lesser *Curvilinear* Figure $abcde$, then those two *Curvilinear* Figures are Similar.



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In like manner having taken two mixt Figures as the two Segments of Circles B A C and $b a c$; and having

Inscrib'd in one any Trian-

gle at Pleasure as $B A C$, if

then you can Inscribe in the

other Segment another Tri-

angle $b a c$, that shall be Si-

milar to the former; then

shall those two *Segments* be

similar Figures. And if the

Circles of which they are Segments be completed, they

shall be similar Parts of those two Circles, so that if BAC

be a third Part of its Circle, $b a c$ shall also be a third

part of its Circle: And if to the Centers you draw

the Lines B D and C D, and also $b d$ and $c d$; the Angles

D and d shall be equal. (See 4. 11. and the following

Propositions.)

50. All Circles are similar Figures.

51. All similar Polygons may be divided into an equal

Number of similar Triangles.

Let the similar Polygons be

A B C D E and $a b c d e$; and

let the first be divided into

Triangles by the Lines E B

and E C (3. 14). I say that if

the other be also divided into

Triangles by the Lines $e b$ and

$e c$, all the Triangles in one

shall be (*respectively*) Similar to those in the other.

For Instance, I say the Triangle $a b e$ is Similar to

A B E: for the Angle a is equal to A (by the supposition)

and also A B. $a b :: A E :: A e$ (by the same); wherefore

the Triangle A B E is Similar to $a b e$. (6. 46). Again,

the Angle E B C may be proved equal to $e b c$; because

the Angle A B C is (by the Supposition) equal to $a b c$,

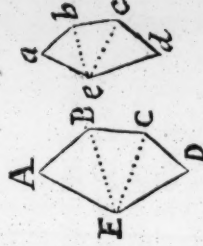
and it was proved (*in the last step, where the Triangle*

A B E was proved Similar to a b e) that the Angle $a b e$

is equal to A B E; wherefore from equal things taking

away equal, the Angle E B C remains equal to the

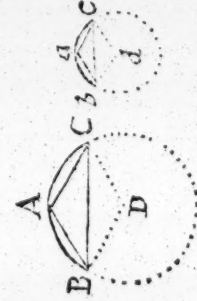
Angle



Angle ebc . In like manner the Angle ecb , is provid equal to ECB , and consequently (6.45) the whole Triangle ebc will be Similar to EBc ; and so of the rest.

52. All similar Polygons are to one another in a Duplicate *Ratio* of, or as the Squares of their Homologous Sides. I say as the Square of AB is to the Square of ab : so is the whole Polygon $ABCDE$ to the Polygon $abcde$. For since all the Triangles in one Polygon, are similar to those in the other (6.51). All in one Polygon will be to all those in the other in a Duplicate *Ratio* of any of their Homologous Sides; that is, as the Square of AB is to the Square of ab .

53. All similar Figures, even Curvilinear ones, are to one another as the Squares



of any side of any similar Figures, which can be Incribed or Circumscribed about them *v. gr.* Let there be two Circles, in which are Inscrib'd two similar Triangles abc and ABC : I say,

the whole Circle ABC , is to the Circle abc : as the Square of BC , to the Square of bc , or which is the same thing, as the square of the Radius BD to the square of the Radius bd . For in or about the Circle abc may be Inscrib'd or Circumscribed any Polygon you please; (or at least such an one may be imagin'd) (4.30). But every Polygon inscrib'd in abc will have a less *Ratio* to the Circle ABC , than the Square of bc hath to the Square of BC : and every one Circumscribed about abc will have a greater *Ratio* to the Circle ABC , as is easie to prove by the Precedent, and from what hath been said of Circles in the fourth Book. Wherefore all similar Figures, &c.

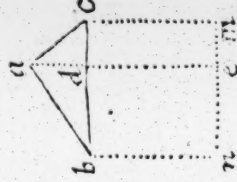
54. All this may be apply'd to Solids. And therefore *Similar Solids* are *such*, as have their Angles all equal, and the sides about those Angles Proportional; or *if they are of a Spherical or of any Spheroidal Figure* *such*,

such, as can have similar Solids Inscrib'd or Circumscrib'd in, or about them, &c.

55. Similar Solids are to one another in a (*TriPLICATE Ratio of, or*) as the *Cubes* (*of their Homologous Sides, &c.*) see 6. 36, 37, &c.

(*And therefore all Spheres must be to one another as the Cubes of their Diameters, &c.*)

56. If in a Rectangle Triangle abc , a Line as ad be drawn from the vertex or top of the Right Angle, Perpendicular to the Base, Hypothenuse, or longest side bc : it shall divide the Triangle abc into two other Rectangled ones, abd and dac , which will be similar to each other, and to the whole bac . For 1. All the three Triangles have one Right Angle. 2. The Triangles abc and abd have the Angle b common to both: Wherefore they are Similar (5. 45). 3. The Triangles abc and dac have also the Angle c common to both; therefore they two are Similar; and lastly abd and dac being both Similar to one third Triangle abc , will be so to each other).



57. The Perpendicular ad is a mean or middle Proportional between bd and dc . That is, $cd. da :: da. db$. For the Triangles dda and bda being Similar (by the last) cd (the lesser Leg of the Triangle dda) shall be to da (the greater Leg) : as the same ad (the lesser Leg of the other Triangle dda) is to bd the greater Leg. (6. 46).

58. The Square of ad is equal to the Rectangle made between cd and db . For, since $cd. da :: da. db$. (by the last,) the Rectangle of the extremes cd and db is equal to the Rectangle of the mean Terms da and da (6. 28). But the two sides of that Rectangle being equal, because 'tis only da taken twice; that Rectangle must be the Square of da ; and so it may be laid down as an universal Theorem, that, &c. (*The Square of the Perpendicular drawn from the Vertex of any Rectangle Triangle*

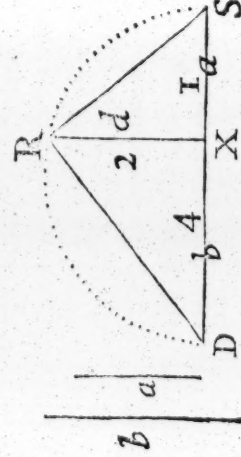
angle to the Hypotenuse, is equal to the Rectangle between the Segments of that Hypotenuse.

59. The Square of a mean Proportional is always equal to the Rectangle of the Extremes.

PROBLEM I.

Between two given Lines a and b , to find a mean Proportional, as d .

Join a and b both in one Line which make the Diameter of a Circle; and then at the Point x , where the given Lines join, erect a Perpendicular as d ; that



shall be the mean Proportional required. For the Angle DRS being a right one (as being in a Semicircle) $b.d : d.a$, by Prop. 57.

PROBLEM II.

And thus may you find a Line equal to the Square Root of any Number or Quantity, by finding a mean Proportional between it and 1. For if $b = 4$, and $a = 1$; then will $d = 2$, equal to the Square Root of b .

PROBLEM

PROBLEM III.

Thus also may a Square be found equal to any Rectangle given, by finding a mean Proportional between its Sides, which shall be the Side of the Square required.

PROBLEM IV.

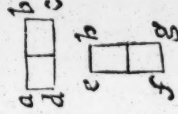
To find a Square equal to any Triangle.

Find a mean Proportional between a Perpendicular let fall from any Angle to an opposite Side, and the half of that Side; and that shall be the Side of the Square required.

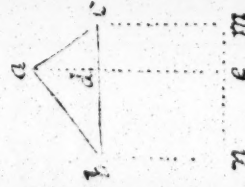
60. *A Rectangle being given to make another Rectangle equal to it, which shall have a length given.*

Let the Rectangle given be ac and let it be required to make another equal to it, the length of one of whose sides shall be the Line ef : here are now three Lines given, viz. ab and bc (which are the sides of the Rectangle given) and ef , which must be one side of the Rectangle required. Therefore a fourth Line must be found, which shall be the other side of the Rectangle sought: Which is done by finding a fourth Proportional to the three given Lines (9. 8). which let be eh . So that $ef. ab :: bc. eh$. and then I say the Rectangle feh is equal to db , and is the Rectangle required. (6. 27).

61. To express a Rectangle you need use but three Letters. viz. gr . When we say the Rectangle bdc : We mean a Rectangle one of whose sides is bd , and the other dc . But if we say the Rectangle bcd , we then mean a Rectangle one of whose sides is bc and the other cd .



61. In every Rectangle Triangle the Square of the Hypothenufe is equal to the (Summ of the) Squares of the two other sides (or to the Summ of the Squares of the Legs).

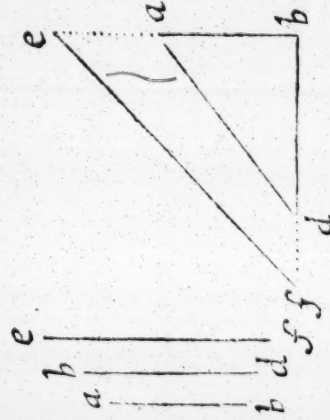


Let the Square $b m$, be divided by the Perpendicular $a d e$ into the two Rectangles $d m$ and $d n$. I say that the Rectangle $d m$ is equal to the Square of $a c$, and the Rectangle $d n$, to the Square of $a b$: and that by consequence the whole Square $b m$ is equal to the Summ of the Squares of $a b$ and $a c$. For 1. The two Triangles $a d c$ and $a b c$ being Similar (5. 56). $d c, c a$ (in the lesser Triangle $d c a$) :: as the same $a c, c b$ in the greater Triangle $a c b$. Wherefore $a c$ is a mean Proportional between $d c$ and $c b$ (or $c m$) and consequently the Square of $a c$ is equal to the Rectangle $b c d$, or $d c m$, that is $d m$.

And after the very samè manner may $a b$ be prov'd to be a mean Proportional between $b d$ and $b c$ (that is $b n$, &c.) (for the Triangle $a b d$ being Similar to $a b c$; $a b$ the lesser Side in one will be to $b a$ the greater side; as that $b a$ (now the lesser side in the other Triangle $a b c$) is to $b c$ the greater side: That is $d b, b a :: b a, b c$. (or $b n$) and consequently the Square of $a b$ is equal to the Rectangle $d b n$, or $d n$. And so both the Squares together, of $b a$, and $a c$, or their Summ, is equal to the Square of the Hypothenufe. Q. E. D.

PROBLEM I.

Hence any two, or more Squares may easily be added together into one Summ. Let $a b, c d$, and $e f$, be the Sides of three given Squares, place $a b$ and $b d$ at Right Angles, and draw the Hypothenufe $a d$, whole Square will be equal to the Summ of the Squares of $a b$ and $a b$. Then set $d a$ from b to e , and the given line $e f$, from b to f ; So shall the Hypothenufe $f e$, be the side



side of a Square equal to the Sum of the three given Squares.

P R O B L E M II.

Or if two Squares be given, you may Subtract one from the other, and find a Square equal to the Difference between them.

Let a and b be the Sides of the given Squares; make (the longest Line) the Radius of a Circle, and let b from the Center on the same Right Line with a ;



at the end of b , erect the Perpendicular p , which will be the side of a Square equal to the Difference between the Squares of a and b the two Squares given; for since the Square of a is equal to the Summ of the Squares of b and p : (by the Precedent Prop.) The Square

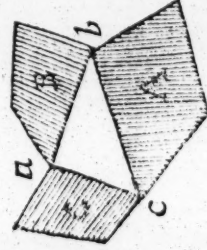
Square of p must be the difference between the two given Squares, whose sides are a and b .

PROBLEM III.

Hence also may a Square be made equal to any given Polygon, or irregular Right-lined Figure: By reducing the Figure into Triangles; finding Squares equal to those Triangles; and then one Square at last equal to the Summ of all those Squares.

Or by making Rectangles equal to those Triangles, which shall have all the same height; then joining those Rect-angles together, so as to make one great one equal to them all; and lastly make a Square equal to that Rectangle.

62. If upon the three sides of a Rectangle Triangle are made three similar Figures, and those similarly Pointed, the greatest shall be equal to the other two.



For the three Figures being similar are as the Squares of their Homologous sides (6. 53). And therefore the Figure A shall be to B and C, as the Square of bc is to the Squares of ab and ac . But the Square of bc is equal to those two Squares (by the last) therefore (the Figure A is equal to both B and C together).

PROBLEM I.

To find two Lines b and c , which shall have the same Ratio to one another, as two given Squares, Triangles, Similar Polygons, or Circles.

Let a and d be the Sides of the two given Squares, Triangles, Polygons; or the Diameters or Radius's of the Circles given: Set them at Right Angles to one

And if from the greater Semicircle, and the two lesser ones you take away what is common to both; which are the two shaded Segments ab and ac ; what remains of each must be equal, *i. e.* the Triangle abc is equal to both the Lunes bna and amc .

And this is the Quadrature of the Lunes of *Hippocrates of Scio*.

64. When the Triangle bac is an *Isoceles*, then the Lunes will be equal, and then also the Triangle abo , being the half of abc , will be equal to each Lune. But if the Triangle be a Scalene as in this Figure, the Lunes are unequal; and 'tis as difficult to divide the Triangle abc into two parts by the Line ao , so as to be able to prove the Triangle abo to be equal to the Lune bna , and the Triangle oac to be equal to the other Lune amc , this is I say, as difficult as to find the Quadrature of the Circle.

N. B. Since this, several ways have been discovered of Squaring any assign'd Portion of these Lunes (See the Philosophical Transactions N. 259, pag. 4. 11). Of which this is one.

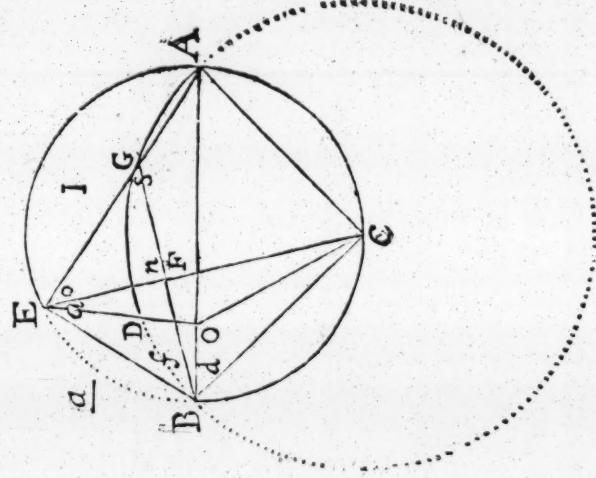
Let there be a greater Circle BGAC on whole Quadrantal Arch BA let the Lune BEAG'B, or L, be drawn by describing the Semicircular Arch BEA, which is one half of the lesser Circle BCAE. Let then a Line as CE be drawn from the Center of the greater Circle cutting off any Portion or Segment of the Lune, as BED: 'Tis required to Square that Segment.

Draw BG at right Angles to EC; So shall the Chord BG be Perpendicularly Bisected in the Point F or n , draw also BE and EGA. I say, That Right-lined Triangle BEF is equal to the Part of the Lune BED.

For FG being equal to FB, EF common to both, and the Angles at F equal because both right, the Triangle EFG will be equal to BEF: Wherefore the Angle

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Angle o being equal to a , they must be both Semi-right: And consequently f and s must be also Semi-right: Therefore the three Triangles EBG , EBF and FBG , must be each one the half of a Square. And consequently $GB.EB::v:2:v:1$. for the Square of GB is double the Square of EB , and since Similar Segments are as the Squares of their Chords, the Segment BG must be double of BE : Wherefore the half of one will be equal to all the other; that is BDF



equal to the Segment BE . And therefore the rectilinear Triangle BEF , exceeding the Portion of the Lune by the half Segment BDF , and falling short of the Lune by the Segment BE , which is equal to that former half Segment BDF , the Triangle is exactly equal to the Portion of the Lune. $Q.E.D.$
And the ground of all is this; that the Angle BCE being at the Center of one Circle, and at the Circumference

lesser which that re- the abc

Hippo-

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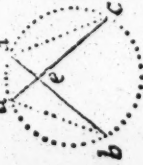
to both, ht, the pre the Angle

ference of the other, must divide the Quadrantal Arch BGA in the same Proportion as it doth the Semicircular one BEA : OA which depends the Equality of the Segments BE and BDE .

And since the Triangle BCA is equal to the Lune L (as is apparent by taking the common Segment $BGAB$ from the Semicircle $BEAB$ and from the Quadrant $BGAC$.) It will be easie to take from thence a part, as the Triangle BOC equal to the Assigned Portion of the Lune. For having let fall a Perpendicular from E to find the Point O , draw OC ; and then will the Triangle BOC be equal to the Triangle BEF , before prov'd equal to the Segment of the Lune. For the Triangles BCA and BEF are Similar, as being each the half of a Square: And therefore the former to the latter will be as the Square of BA . to the Square of BE (6. 47) their Homologous Sides. That is, as BA . is to BO (6. 30) for BE is a mean Proportional between BA and BO . Farther, the Triangle BAC , having the same height with BCO , will be to it as the Base AB to BO . Wherefore the two Triangles AEF and BCO , being proved to have the same Ratio, to one and the same thing must be equal. $Q. E. D.$

And therefore to divide the Lune according to any given Ratio, you need only divide the Diameter AB according to that Ratio in the Point O , and from thence erect a Perpendicular to find the Point E ; then draw EC , which shall cut off the assigned Portion of the Lune.

65. Two Chords cutting or crossing each other in a Circle have their Segments *Reciprocally Proportional*.



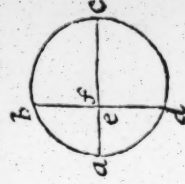
I say that $ae.be :: de.ec$. and consequently the Rectangle aec is equal to the Rectangle deb .

For draw the prickt Lines ab and dc and the two Triangles abe and dce will be Similar; Because 1. The vertical or opposite Angles at e are equal (1. 23). 2. The Angle

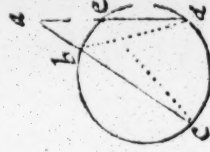
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Angle b is equal to c , because standing both on the same Ark ad ; and being in the same Segment (4. 12). wherefore the two Triangles are Similar, and consequently $ae.eb::de.ec$. (6. 46). Q. E. D.

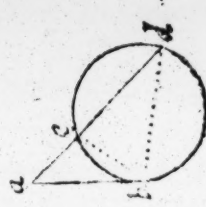
66. If ac be the Diameter of a Circle, and bd , a Perpendicular to it, de or be will be a mean Proportional between the Segments of the Diameter ae and ec . Because de is equal to eb (by 4. 6), and therefore since (by the last) the Rectangle bed (that is be Square) is equal to aec , as the Rectangles of the Parts of all crossing Chords are; the Line be or cd must be a mean Proportional between ae and ec . Q. E. D.



67. Two Lines ac and ad drawn from a Point a without a Circle, to the internal and opposite Part of its Circumference; are to each other *Reciprocally* as their external Segments. I say, $ac.ad::ae.ab$. and consequently the Rectangle cab is equal to dae . For, supposing the Lines ce and bd to be drawn, the Triangles ace and adb will be Similar, because the Angle a is common to both, and the Angle c is equal to d because standing on the same Ark be (4. 12). wherefore $da.ab::ca.ae$, and Alternately $da.ca::ab.ae$, and by Inversion $ca.da::ae.ab$ (6. 45). And therefore the Rectangle cab is equal to dae . Q. E. D.



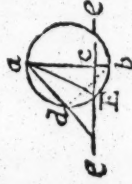
68. If one Line as ab touch a Circle, as in the Point b , and another Line ad drawn from the same Point a , do cut it; Then is ab (the Tangent) a mean Proportional between ad and ae (i. e. between the whole Secant and the Part of it without the Circle.)



For, drawing the Lines be and bd , the Triangles aeb and bda will be Similar; Because the Angle a is common to both, and the Angle abe (made by the Tangent, and Secant eb)

is equal to d (*an Angle in the opposite Segment*) (4. 17). therefore they are Similar; And consequently $e a$ (in the little Triangle) will be to $a b ::$ as that same $a b$ is to $a d$ in the greater Triangle: i. e. $e a. a b :: a b. a d$. Q. E. D.

69. Let there be a Diameter ab cut in c by an Infinite Perpendicular ee , whether within the Circle as in *Fig. 1.* at the Circumference as in *Fig. 2.* or without the Circle as in *Fig. 3.* Let there be drawn also from the Point a any Right Line as ae cutting the Perpendicular in e and the Circle in d . I say it shall always be as $ad.ae :: ab. ae$.

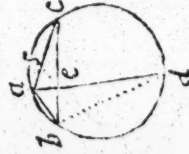


For drawing the Line bd , there will be made two Triangles that are Similar, as $eaec$ and dab ; which will be so, because they have one Angle as a common to both; and the Angle d equal to e , because both are Right ones (for d is Right by 4. 14), as being an Angle in a Semicircle and e is Right by the Supposition. Wherefore the Triangles are Similar and Consequently $ad.ae :: ab.ae$. Q. E. D.

70. In the second Figure ab is always a mean Proportional between ae and ad ; and in the first, the middle Proportional is aE , drawn from a to the place where the Line ec cuts the Circle.

71. If of a Triangle inscrib'd in a Circle, the Angle $b ac$ be Bisected by the Line $ae d$.

I say, then $ba.ae :: ad.ac$. For drawing the Line bd there will be made two Triangles abd and aec , which are Similar; because the Angle d is equal to c (4. 12), as (*being in the same Segment*) or insinuating on the same Ark, and bad is equal to $eaec$ by the Supposition. Wherefore the Triangles are Similar, and Consequently $ba.ad :: ad.ac$.



Q. E. D.

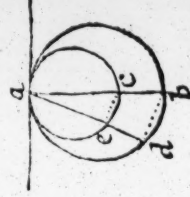
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72. When the Angle at the Vertex is thus Bifected, the Segments of the Base bc , are also Proportional to the Legs of the Triangle (*i.e.*) $be.ec::ba.ac$. For supposing ef drawn Parallel to ba . Then will $ba.ac::ef.fc$ (6. 40). But ef is equal to af : because the Angle aef is equal to eab , (*as being Alternate Angles* 1. 31), and consequently to ef (*by the Supposition*) wherefore the Triangle aef is an *Ifoſceles* (2. 15). And therefore inſtead of putting of it as before $ba.ac::ef.fc$, we may ſay $ba.ac::af.fc$. But as $af.fc::fo$ is $be.ec$ (6. 42). wherefore $ba.ac::be.ec$. Or, which is all one $be.ec::ba.ac$. Q. E. D.

(N. B. This Proposition is Universal; and if any Angle of a Triangle be Bifected, the Legs about that Angle are Proportional to the Segments of the oppoſite ſide made by the Line Biſecting the Angle.)

73. If two Circles touch one another (*in a Point with- in*) as a , and if to that Point you draw a Tangent and a Perpendicular aeb (which will paſs through both their Centres) (4. 5) and if alſo you draw any Secant from the ſame Point as aed . I ſay, 'twill always be as $ae.ad::ac.ab$. For having drawn the Lines ec and db , the Triangles aec and adb will be Similar, as having the Angle at a , common; and e and d both Right ones; (by 4. 14). and conſequently $ae.ad::ac.ab$. Q. E. D.

74. The Ark ec is to the Ark db as the whole Circle aec to the whole Circle adb . (6. 49 and 4. 11).



BOOK VII.

Of Incommensurables.

1.

A Lesser Quantity is said to *Measure* a greater, when being taken a certain number of times, it is exactly equal to the greater. *v. gr.* Suppose a Fathom to contain six Feet; then may one Foot be said to *Measure* that Fathom, because being taken or repeated six times, it will be exactly equal to the Fathom.

2. The Quantity which is thus a *Measure* to a greater Quantity, is call'd a *Part* of that greater; and the greater Quantity is call'd the *Multiple* of the lesser. So, a Foot is the *Part* of a Fathom, and a Fathom is the *Multiple* of a Foot.

3. If you take the Quantity (*of a common French Pace*) which is two Foot and a half, and try with that to *Measure* a Fathom, you cannot do it: Because if you add that Pace only twice, it will make but five Foot, which are less than the Fathom; and if you take it three times, it makes seven Foot and a half, which are more than the Fathom; so that this Quantity of two Foot and an half cannot *Measure* the Fathom, and therefore properly speaking is not a *Part* of it. But nevertheless they may be said to be *Parts* of the Fathom, because this Quantity contains five half Feet; for an half Foot is a *Part* of a Fathom, because being taken 12 times it will *just Measure* it; so therefore this Pace contains *Parts* of the Fathom, because it contains five half Feet which are $\frac{5}{12}$, that is five twelfths of a Fathom.

4: When

4. When two Quantities are such, that a third can be found which shall be an (*Aliquot or Even*) Part of both, that is, which shall *Measure* them both exactly: Then those Quantities are said to be *Commensurable*; As for instance, a Pace and a Fathom are two Commensurable Quantities, because we can find a third Quantity, *viz.* half a Foot, which will Measure them both; For if the half Foot be taken five times it makes the Pace, and taken twelve 12 times, it makes the Fathom.

5. But when it is not possible to find any third Quantity which can Measure two others, then those two Quantities are call'd Incommensurables.

6. Commensurable Quantities are as *Number to Number*, that is, those Quantities can be expressed by Numbers, so that as one Quantity is to the other, so shall one Number be to the other. Thus a Line of six Feet or a Fathom, and a Line of two Foot and an half, as a Pace, are to one another as Number to Number. For half a Foot Measuring them both, the latter by being taken 5 times, and the former by being taken 12 times; it's plain that one Line contains 5 half Feet, and the other 12, and therefore they are as 5 to 12, or as Number to Number.

7. If two Quantities are not as Number to Number, that is, if it be impossible to express their Magnitudes by two Numbers, they are Incommensurable: As is plain from the last.

8. We ought then to see whether there are in Reality any such Quantities whose Magnitudes cannot be express'd by Numbers, and if there be any *such*, we must say that there are *Incommensurable Quantities*.

9. A *plane Number* is that which may be produced by the Multiplication of two Numbers (*one into another*). *v. gr.* 6 is a plane Number, because it may be produced by the Multiplication of 3 by 2: For twice 3 makes 6: So also 15 is a plane Number arising from 5 being multiplied by 3; and 9 is a plane Number produced by the Multiplication of 3 by 3.

10. Those Numbers which being multiplied one by another, do produce a plane Number, are call'd the *Sides* of that Plane, as 2 and 3 are the Sides of the Plane 6; and 3 and 5 are the Sides of 15.

11. If we imagine Unites to be little Squares, those Squares may be form'd into a Rectangle, if their Number be a Plane. *v. gr.* 12 Squares may be placed in the form of a Rectangle, one of whose sides may be 6 and the other 2. and 48 will make a Rectangle whose two sides may be 12 and 4. See the following Figures B and C.

12. A *Square Number* is a plane, whose sides are equal; as 4, arising from the Multiplication of 2 by 2; as 9, the Product of 3 by 3; and 16 made by 4 multiplied by 4, &c.

13. A Square Number may be rang'd into the form of a Square and that Number which can be ranged into the form of a Square is a Square Number, and that which cannot be rang'd into the form of a Square, is not a square Number.

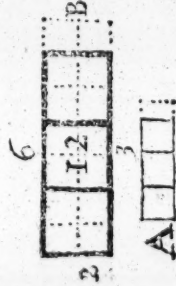
14. *Similar plane Numbers* are those which may be ranged into the form of similar Rectangles; that is into Rectangles, whose Sides are Proportional; such are 12 and 48; For the Sides of 12 are 6 and 2 (*See Fig. B*) and the Sides of 48 are 12 and 4 (*See Fig. C*). But 6. 2 :: 12. 4.

and therefore those Numbers are Similar.

15. All square Numbers are similar Planes (6. 32).

16. Every Number may be placed in the form of a Right-line, and in that disposition may be taken for a Plane.

Thus 3 (*in Fig. A*) may be conceived as a Plane Similar to 12 or B; For the sides



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sides of the Plane 3, are 1 and 3, (because once 3 is 3) and the sides of 12 are 2 and 6. But as 1. 3 :: 2. 6.

17. There are Numbers which are not *Similar Planes*.

As if you examine from 1 to 10, you will find indeed that 1. 4. 9. being Squares are Similar, and so are 2 and 8, which have one side double to the other. But the rest as 3. 5. 6. 7. are by no means similar Planes.

18. If one square Number be multiplied by another, the Product will be a third square Number.

Thus A. 4. and B. 9. being both Squares do, when multiplied into one another, produce the Number 36 or C: And I say that third Number is a Square. For the meaning of

multiplying B by A, is to take B as often as there are Unites in A. But

I may consider the whole Number B. 9. as one only Square, and I can

take that as often as there are Unites, or little Squares in A. And as the

rang'd into a Square, so I can range the Square B as often into a Square Form, just as if it were an Unite. So

that there will be four such Squares of B, which being placed as you see in the Figure, will make the Square C or 36.



19. If two Numbers are similar Planes, the greater may be divided into as many Squares, as their are Unites in the lesser. A. 3. and B. 12.

are similar Planes; so that the

side 3. is to 6 :: as the side 1.

is to 2. Wherefore I can divide

the Plane B. 12. into 3 Squares

placed just in such manner as

those 3 little Squares in the

Plane A. And every one of

the great Squares of B shall answer to 4 of those in

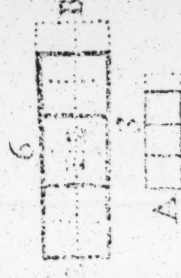
A. So also if the Planes had been 8 and 72; I can divide

72 into eight Squares, of which every one shall contain

9 of those in the lesser Plane 8. The same would come

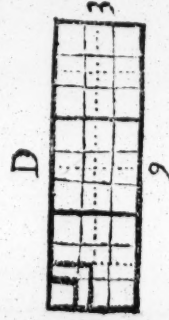
to pass also, if either one, or both the Numbers had

been



been Fractions. As if A contain 3 and $\frac{1}{2}$, and B. 14. I can divide 14 into three Squares and an half, disposed

just like those in A. as may be seen by the Partitions in the Figure, and by the half Square added in Prickt Lines.



In like manner if the

Planes were B 12, and D 27,

I can divide 27, not only into three Squares, dispos'd after the same manner as those in A: But also in to 12 Squares, so ranged as those in B. as the prickt Lines in the Figure D do shew. The way to do which, is to divide the sides of the greater Plane into as many Parts as the Homologous sides of the lesser Plane are divided into; the Figure shews the thing, and makes it ealie.

20. Those plane Numbers which can be so divided as that there are as many Squares in the greater Plane, as there are Unites in the lesser, are Similar; this is the Converse of the former.

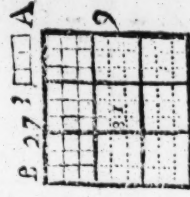
2. Two similar Plane Numbers multiplied one into another do produce a Square. For having divided the greater Plane into as many Squares as there are Unites in the lesser (7. 19.) One Plane will be multiplied by the other, if the greater Squares of the greater Plane be taken as often as there are Unites or little Squares, in the lesser Plane: But to a multiply any Number of Squares, by the same Number, is to make one Square out of all those Squares.

For Instance, A 3. and B 27. being Similar Planes, I consider B. 27 as a Plane compos'd of three great Squares, as A 3. is a Plane compos'd, of three Unites, or three little Squares. So that if I take all these three great Squares, as often as there are Unites in A, that is three times; I produce then three times three such great Squares as are in B, that is, 9 such Squares; of which every one contains

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9 of those in A, and all these 9 Squares of B contain 81 of those of A; so that A 3. multiplying B 27. produces 81. which is a Number of

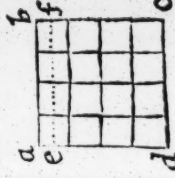
the lesser Squares rang'd into a Square Figure; and by consequence a square Number (7. 13). In like manner, if the Planes were B. 12. and D. 27. I divide 27 into 12 Squares, which I multiply by 12, and there are Produced 144 greater Squares rang'd into the form of a Square, which do contain in all 324 of those of the lesser Plane. (N. B. To divide 27 into 12 Squares, each Square must be 2. 25. (or two and a quarter) as you see it is in the Figure D. N. 19).



22. If two Plane Numbers are Similar, after what form soever you range one, the other also may be so dispos'd. Let 3 and 12 be similar Planes. If 12 be so rang'd in a Right-line that it will make a Rectangle, one of whose sides shall be 12, and the other 1. I say that 3 may be so dispos'd as to make a similar Rectangle, one of whose sides will be 6, and the other the half or one, &c.

23. If one Number divide another that is a Square one, a third shall be produced which will be a Plane Similar to the Divisor.

Let there be a Square ac 16, and let it be divided by any Number, as suppose by 8. which is done if you take the eighth Part of the side ad , viz.



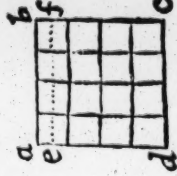
ae , and thro' e draw the Parallel ef . For by that means you will have the Plane af , which will be the eighth Part of the Square ac . But to divide a Number or a Plane by 8, is to take the eighth Part of that Number or Plane.

I say the Plane af , is similar to 8; for 8 being rang'd into a Right-line, so as to make a Rectangle, one of whose sides shall be 8. and the other 1. shall be Similar to it, because ae was taken the eighth Part of ad or ab : Where-

another Right-line as *AD*. and that out of those two Right-Lines there be formed the Plane *AC*. 16. That Plane *AC*.

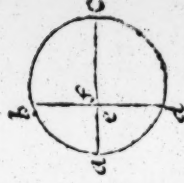
16. will be produced by the Multiplication of the two Numbers 2 and 8 (6. 17 and the following Propositions) and consequently the Number of the little Squares of the whole Plane *AC*. 16. shall be a square Number (7. 21) and they may be rang'd into the form of a Square (7. 13).

Let them then be disposed into the Square *ac*. So shall the Square *ac* be equal to the Plane *AC*. for 'tis on- ly the same Number dispos'd or rang'd after another manner. Wherefore (6. 59) the side *ab* 4 shall be a mean Proportional be- tween *AD* 2. and *AB* 8.



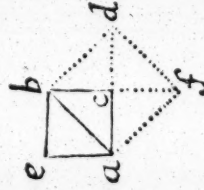
29. Between two Numbers Non-similar a mean Pro- portional can't be found. Let the Numbers be 4. and 6. Range each of them into a Right-line, and multiply them, they will produce the Plane 24. But this Plane 24 is not a square Number (7. 25). and consequently cannot be rang'd into a square Form. Wherefore 'tis impossible to have any *Mean* between 4. and 6. For such a pretended mean Proportional must, multiplied by it self, produce a Square, which (as hath been prov'd elsewhere) will be equal to the Plane made between 4. and 6. (6. 59). which is impossible, because this Plane 24, made out of 4 and 6. is not a square Number.

30. Let there be two Lines *ae* and *ec*, so to one ano- ther, as one Number to another Non- similar. *v. gr.* as 1. to 2. Let also *eb* be a mean Proportional, so that *ae. eb :: eb. ec*. I say that *eb* is *Incommensurable* with the two Extremes *ae* and *ec*. For *ae* and *ec* being as 1. to 2. (*i. e.*) as Numbers Non-similar (by the Supposition) as also are all their Equi- multiples (7. 27), 'tis impossible to find a mean Propor- tional



tional between ae and ec (by the Precedent) and consequently, eb cannot be to ae , or to ec , as Number to Number. Wherefore it is Incommensurable with them.

31. The Diameter of a Square ab is Incommensurable to the side ac . For taking ad double to ac , and making the Triangle abd , it shall be Similar to the Triangle abc ; because cd being equal to cb , the Angle d is equal to cdb (2.15) and the Angle d must be a Semi-right one as well as cab ; wherefore abd is a Right-angle; and consequently $ac, ab :: ab, ad$. That is, ab, ac is a mean Proportional between $ac, 1$. and $ad, 2$. and therefore Incommensurable (by the Precedent).



C O R O L L A R Y.

Hence 'tis impossible to Express the Double of a Square in Rational Numbers.

32. The Power of a Line is the Square, which is made upon it. Thus the Power of the Line ac (Fig. *preced.*) is the Square $aebc$; and the Power of the Line ab is the Square $abdf$. And we say that Line ab is double in Power (in Latin *bis potest*) to the Line ac , which is a manner of Speaking borrowed from the Greeks, and generally received amongst Geometers.

33. The Diameter ab is *Commensurable in Power* to the side ac : That is, it's Square $abdf$ is Commensurable to the Square $aebc$; for 'tis indeed double to it.

34. But if you take ao , a mean Proportional between ab and ac , that mean ao shall be Incommensurable to them even in Power; i.e. the Square of ao is Incommensurable to the Square of ab , or to the Square of ac . for the Square of ac to the Square of ao . is in a Duplicate Ratio of ac to ao (6.29); that is, as ac to ab (6.30). But ac is Incom-

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Incommensurable to ab (7. 31). wherefore the Square of ac is Incommensurable to the Square of ab .

35. There is a *Second Power* of a Line which is call'd the Cube, which is made by multiplying the Square, by that first Line, or Root.

36. If two mean Proportionals an and am be taken between ac and ab ; so that $ac.an::am.ab$.

the Line an will be Incommensurable $a \text{ --- } n$ in this second Power to ac (*i.e.*) The Cube of ac will be Incommensurable to the

Cube of an , because the Cube of ac to the Cube of an is in a *Triplicate Ratio* of the side ac to the side an ; *i.e.* as ac to ab . But ac and ab are Incommensurable, wherefore, &c. However ac and am are Commensurable in the second Power, for the Cube of am is double to the Cube of ac .

37. 'Tis easie to apply to *Solid Numbers* what hath here been said of Plane ones. And those are call'd *Solid Numbers* which arise from the Multiplication of a Plane Number by any other whatsoever. *v.gr.* 18 is a solid Number made of 6 (which is a plane) multiplied by 3; or of 9 multiplied by 2.

38. *Similar Solid Numbers* are those, whose little Cubes may be so rang'd as to make similar and rectangular Parallelopipeds.

39. *Cubick Numbers* are such as can be rang'd into the form of Cubes as 8, or 27, whose *sides* are 2 and 3 and their *Bases* 4 and 9.

40. Every Cubick Number multiplying another Cubick Number produces a third Cubick Number.

41. Between two similar solid Numbers there may be found two mean Proportionals.

That which hath been demonstrated in respect to Plane Numbers, may be applied to Solids.

42. These Demonstrations by which 'tis prov'd that there are Incommensurable Lines and Magnitudes, shew also that a *Continuum* is not Compos'd of Finite Points: For if the Diameter as well as the side of a Square were

were compos'd of Finite Points, a Point would measure both the side and the Diameter: for that Point would be found a certain Number of times in the side and another determinate Number of Times in the Diameter, which the preceding Propositions prove impossible.

43. Because in a Rectangle Triangle the Square of the Hypotenuse is equal to the Summ of the Squares of the Legs; (6. 61). we have always used this Triangle for the discovery of Incommensurables. For if all the three sides are Commensurable, they may be all three express'd by three Numbers, and then the Square of the greatest Number will be equal to the Summ of the Squares of the other two. As if the greatest side or Hypotenuse be 5 Feet, the least side three, and the middle one 4: The Square of 5 will be 25; the Square of 3, 9; and the Square of 4 will be 16: And 9 and 16 added together do make the great Square 25. But if the least side of such a Triangle be 2. and the middle one 3. then the greatest side cannot be express'd in Numbers, because the Square of the least side 4 added to the square of the middle side 9 make 13, which expresses the Square of the greatest side. But as that Number 13 is not a square Number, so its side or Root cannot be express'd by any Number.

44. At all times Men have been Sollicitous to find out some Method of discovering proper Numbers to express the three sides of a Rectangle Triangle, so as to be assured that all the three sides are Commensurable. Therefore I here shew you such a Method, by which you may find out all the possible Numbers that are proper for this purpose.

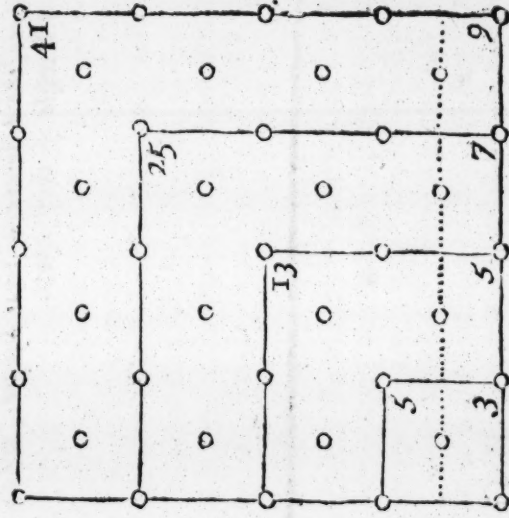
45. If you take any two Numbers (even Unity itself) differing but by an Unite, and add the Squares of them together, the Summ will be a Number which shall be the Root of a Square equal to two Squares; And that Number will express the greatest side of a Rectangle Triangle, whose middle side shall be that Number less'n'd by Unity, and the least side shall be the Summ of the two first Numbers. *v. gr.* Having taken 1 and 2; and

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and Squared each of them you have 1 and 4; Add those two Squares together, and the Sum is 5. I say 5 will express the greatest side; and then 4 will be the middle one and 3 the least; and 25 the Square of the Hypothenuse will be equal to the Sum of the other two Squares. In like manner if you take 2 and 3, and add their Squares 4 and 9 together, the Sum is 13. Then I say will 13, 12 and 5 be three sides of a Rectangle Triangle; so that 169 the Square of 13, shall be equal to 144 and 25, the Squares of 12 and 5. Moreover if you take 3 and 4: the Sum of their Squares 9 and 16, makes 25; wherefore I say 25 may be the greatest side of a Rectangle Triangle, whereof 24 will be the middle side and 7 the least side.

All which may more easily be found after this manner.

46. If you range Unites into a square Form as in the Figure; all those Numbers which make a square Figure



are proper Numbers to express the greatest side; the least side shall be the Number contain'd in the two first Ranks of that square Figure, and the middle side shall be a Number less than the greatest, by one, or Unity. H 47. If

47. If this Figure were continued it would give you all possible Numbers; But it must be observ'd also that the Equi-multiples of any 3 Numbers thus found will do the same thing: Thus, having found 5, 4 and 3 their doubles 10. 8. and 6. will represent the three Sides of a Rectangle Triangle, so that 100 the Square of 10 shall be equal to the Sum of 64 and 36 the two Squares of 8 and 6. And their Triples also 15. 12. and 9. will do the same thing: For any one may see that all these Numbers still having the same Proportion, do as it were constitute but one only Triangle, viz. that which is express'd by 5. 4. and 3. And therefore all those Numbers may be taken for the same.

N. B. The three Sides of a Rectangle Triangle will then only be Commensurable, when they are in this Proportion, viz. as $aa + ee$, $aa - ee$ and $2ae$. That is, the Summ of two Square Numbers, the difference of their Squares, and the double Rectangle of their Roots.

BOOK VIII.

Of Progressions and Logarithms.

1. **P**rogression is a *Series* or Rank of Quantities which keep between one another any kind of similar Relation or Proportion; and every one of these Quantities is called a *Term*.

2. When the Terms which so follow one another do equally Increase or Decrease, the Progression is called *Arithmetical*; as are all Numbers proceeding according to the natural Order of the Figures, as 1, 2, 3, 4, 5, 6, &c. As also all odd Numbers, as 1. 3. 5. 7. 9. 11, &c. or as 4. 8. 12. 16. or as 20. 15. 10. and the like.

3. Arithmetical Progression may be increased Infinitely, but not diminished.

4. If in an Arithmetical Progression there be four Terms, the Difference between the two first of which is equal to the difference between the other two, those four Terms are said to be *Arithmetically Proportional*: As in the Progression of the natural Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. If you take four as 2. 3 :: 9. 10 (*This mark :: shall for the future use to signify Arithmetical Proportion*) there will be the same Arithmetical Proportion between 2 and 3 as there is between 9 and 10; that is, 10 exceeds 9, as much as 3 doth 2: so also 3. 5 :: 8. 10 are in Arithmetical Proportion; and so are 1. 5 :: 5. 9. where 5 being taken twice, is an Arithmetical mean Proportional between 1 and 9.

5. In Arithmetical Proportion the Aggregate or Summ of the two extrems is equal to the Aggregate of the two Means, as in $2. 3 :: 9. 10$. the Summ of 2 and 10 is equal to the Summ of 3 and 9, that is 12; so also in $3. 5 :: 8. 10$. The Aggregate of 3 and 10 is 13, which is equal to the Aggregate of 5 and 8. And the reason of this is self-evident. For tho' 10 exceeds 8, yet that which is added to 8 (*viz.* 5) doth just as much exceed 3, which is added to 10, and so there necessarily arises an Equality between them.

6. The Summ of the first and last Terms in any Arithmetical Proportion is equal to the Summ of the second and the last save one; or to the Summ of the *third from the first Term, added to the third, accounted backward from the last*, &c. as in the first Example, 1 and 9 make 10; and so do 2 and 8, 3 and 7, or 6 and 4 always make 10. And in the middle remains 5, which being taken twice (as if it were equivalent to two Terms, because 'tis equally distant from the first and last Term) makes also 10.

7. If you add the first Term to the last, and multiply that Summ by half the Number of the Terms, the Product shall be equal to the Aggregate or Summ of all the Terms. As in the former Example, 1 added to 9, makes 10, and 10 multiplied by $4\frac{1}{2}$ (or 4, 5) for there are 9 Terms, produces 45 which is the Summ of all the Terms from 1 to 9. As is manifest from the Precedent.

8. When the Terms of the Progression are continual Proportionals; that is, when the first is to the second :: as that is to the third Term :: as the third is to the fourth, and as the fourth is to the fifth, &c. then the Progression is call'd *Geometrical* as 1. 2. 4. 8. 16. 32. &c. Or as 1. 3. 9. 27. 81 $\div$ or again, as 3. 12. 48. 192. 768. or *descending*, as 8. 4. 2. 1 $\div$; or lastly as $\frac{1}{2}$. $\frac{1}{4}$. $\frac{1}{8}$. &c.

9. Geometrical Progression may be encreas'd and diminish'd Infinitely.

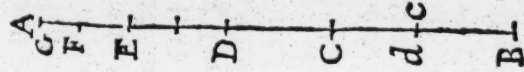
10. When

10. When the Progression begins with 1, the second Term is call'd the *Root*, *Side* or *first Power*; the *third*, is call'd the *Square* or *second Power*; the *fourth*, the *Cube* or *third Power*; the *fifth*, the *Biquadrate* or *fourth Power*; the *sixth*, the *Solid* or *fifth Power*; the *seventh* the *Quadrato Cube* or *sixth Power*, &c.

11. If (in such a Progression) you take four Terms, the former two of which are as much distant from each other, as the two latter are: Those are simply Proportional, and the Rectangle of their extremes is equal to that of their two middle Terms.

12. Let the Quantity AB be so divided in C, D, E, and F, &c. that AB. AC :: AC. AD :: AD. AE, &c. Then I say BC. CD. DE. EF, &c. are in continual Geometrical Proportion; and also that AB. AC :: BC. CD :: CD. DE, &c. for because AB. AC :: AC. AD. it will follow by Division of Proportion, that AB less AC (that is CB) AC :: as AC less AD (that is DC) AD. and consequently Alternately CB. CD :: AC. AD :: or as AB. AC. and so of all others it may be proved :: DC. ED :: EF :: GF, &c.

13. Let there be a Progression of Quantities in a Right-line BC, CD, DE, EF, &c. let Cd be equal to the second Term CD, that so we may have Bd the difference between the first and second Terms: And let it be made as Bd. BC :: BC. to a fourth Line, viz. BA. I say that, if the Number of the Terms BC. CD. DE, &c. be Finite, tho' never so great, all those Terms taken together, although there be an hundred thousand Millions of them, shall be less than BA. But if we suppose the Progression infinite, or that the Terms are Infinitely many, then shall all of them taken together be exactly equal to BA. For since by the supposition Bd. (that is BC less Cd or CD) is to BC :: BC. (that is AB less AC) AB. it may easily be found that as BC. CD :: AB. AC :: AC. AD, &c.



&c. and consequently all the Terms $CD, DE, EF, \&c.$ will always be found within, or be hither the Point A . To which it Approaches the nearer, the more the Number of the Terms is increas'd. So that we see plainly, that all these Terms (which in Books are usually call'd *Parts Proportional*) tho' they be actually Infinite, cannot make an Infinite length, because they will all be included within the Line BA .

14. This Demonstration will appear much more easie and sensible by the Example of a particular Progression, where the Terms are in a double *Ratio v. gr.* Let CB be double to DC and DC double to DE , &c. For if the Number of the Terms be here Finite, tho' it be an hundred thousand Millions, and you take the last and least Term, for Example F , and add to it another Quantity, as suppose A , equal to it: It is then plain that E must be equal to the Term ED , which is the last save one; For ED is double to E by the Supposition (the *Ratio* being every where double) and E is also double to E by the Construction, it having been made so, by taking F equal to E . In like manner A with DE , that is A , shall be equal to the following Term CD , and at last A will be equal to BC . So that from hence it appears, that the first or greatest Term is always equal to all the others taken together, provided there be added to them but a Quantity equal to the last and least Term; but if nothing be added, the first Term is always greater than the Sum of them all.

If these Terms are suppos'd to be actually Infinite then the greatest BC will be exactly equal to all those Infinite others taken together $CD, DE, EF, \&c.$ For any one may easily discern, that the more there are of such Terms, the more you approach towards A . by cutting off still the half of the remainder: But when any Quantity is thus lessen'd by half, and the Remainder again by half, and then the half of that third Remainder taken and so on: 'Tis plain that by supposing the Diminution to be made an Infinite number of times, nothing at last will Remain.

This

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This also might be demonstrated by a *Reduction ad impossibile*, by shewing that all those infinite Terms are, taken together, neither greater nor less than B A.

15. Hence may the Difficulties raised by the School-men against the (*Infinite*) Divisibility of a *Continuum* be solved, tho' to persons Ignorant of *Geometry* they appear Unsolvable: But indeed at the Bottom, they are nothing but meer Paralogisms.

16. If two Progressions are supposed, one Geometrical beginning with 1. and the other Arithmetical beginning with 0, so that the Terms in one shall be Placed over and answer Respectively to those in the other. The Arithmetical ones are call'd *Logarithms*, *Exponens*, (or *Indexes*) as in the following Ranks.

0.	1.	2.	3.	4.	5.	6.	7.	8.
1.	2.	4.	8.	16.	32.	64.	128.	256.

17. That which is produced in a Geometrical Progression by Multiplication and Division; is effected in the *Logarithms* by Addition and Subtraction: As, if having three Numbers given: $2.8::64$; You would find a fourth Proportional to them in Geometrical Progression: You must multiply 64. by 8. (which are the two middle Terms) For the Product, 512. shall be equal to the Product made by 2. and the fourth Number sought, they being the two extremes of four Proportionals. And to find this fourth Number, you need only divide 512 by 2, and the Quotient will be 256. So that $2.8::64.256.$ and 64 and 256 will be just as far distant from one another in the order of the Progression, as 2. and 8 are.

But if instead of the Geometrical Numbers $2.8::64$ you had used their Logarithms $1.3::6$, which Answer to them in the Progression, and were minded to find a fourth Logarithm, then you must have added 3 and 6 which make 9, and from thence have Subtracted 1. there would remain 8. The Logarithm answering to the Geometrick Number 256.

H 4

18. So

18. So also, if there be two Geometrick Numbers 4 and 8, to which the Logarithms 2 and 3 do answer; by multiplying 4 by 8 you produce 32; the Number under the Logarithm 5, which is the Sum of the Logarithms of 2 and 3.

16. In like manner by multiplying 16 by it self, there will be produced 256 which stands under the Logarithm of 8, the Sum of 4 added to it self.

20. So if the Geometrical Number were required that shall answer to or stand under the Logarithm 16; you must take 256 which stands under 8, and multiply it by it self, and it will produce 65536 the Number requir'd.

21. If moreover the Geometrical Number answering to the Logarithm 23 were required, you may take any two Logarithms whose Sum is 23, as suppose 7 and 16 and multiplying the Geometrical Number under them viz. 128 and 65536 one by another, the Product will be 8388608. The Number which ought to stand under the Logarithm 23, or in the 23 place of a Series of Geometrical Proportionals beginning from 1.

22. From hence appears the way of Answering that ordinary Question, how much a horse would cost, if bought on this Condition: That for the first Nail in his Shoe a Farthing were to be paid, for the second Nail two Farthings, for the third Nail four Farthings, for the fourth Nail eight Farthings, and so on, still doubling for 24 times: For the 24th place in such a Progression would be the last Number 8388608 Farthings, which being reduced is 8738 l. 2 s. 8 d. and being doubled according to (8. 14). gives the whole price of the Horse 17476 l. 5 s. 4 d.

23. Where two compleat Progressions are fitted so as to answer one to another, the Geometrical to the Arithmetical; as suppose in Tables for that purpose calculated in Books, there abundance of Pains and Labour is spared, in finding the Geometrical Numbers: For Instance, let those three Numbers 32. 64. 128. be given, and that a fourth Proportional were required:

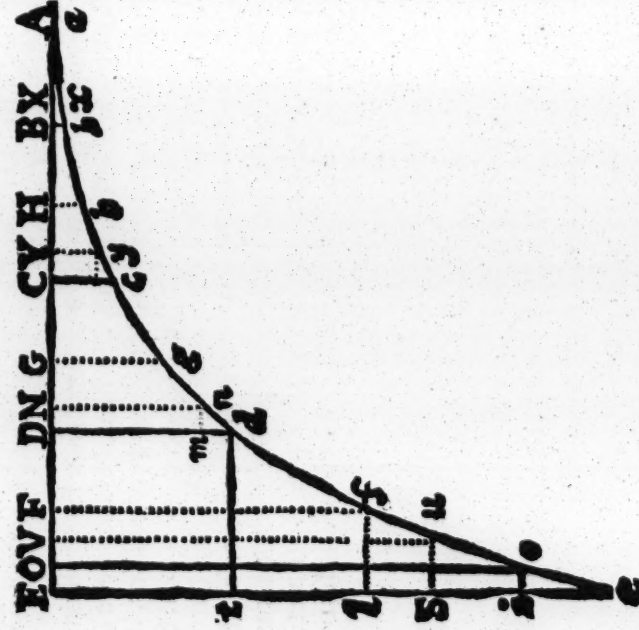
Instead

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Instead of multiplying 64 by 128, and dividing the Product by 32 (which way is very tedious in great Numbers): you need only take the Logarithms of the three given Numbers, viz. 5, 6, 7. and adding 6 and 7 together the Sum is 13, from whence substract 5. there rests 8 which is the Logarithm of the fourth Geometrical Proportional sought, seek therefore in the Table for the Logarithm 8. and the corresponding Geometrical Number you will find to be 256.

24. But because in such a Geometrical Progression all Numbers will not be found, this Medium hath been discovered; they have calculated two Progressions, one of which, contains all Numbers 1. 2. 3. 4. 5. 6. 7. 8. &c. which seems to be an Arithmetical Progression, but yet hath in reality the Properties of a Geometrical one. And the other which contains Numbers in appearance the most Irregular, is nevertheless a true Arithmetical Progression. See here a Line, which will discover perfectly all these Mysteries.

25. Let the Right-line AE be divided into the equal Parts AB, BC, CD, DE , &c. from the Points A, B, C, D, E , &c. let the Lines Aa, Bb, Cc, Dd and Ee , be drawn all (*perpendicular to AE and consequently*) Parallel to one another : And let them be all in a Geometrical Progression : As, let Aa be 1, Bb 10, Cc 100, Dd 1000,



Ee 10000, &c. Then shall we have two Progressions of Lines, the one Arithmetical and the other Geometrical : For the Lines AB, AC, AD, AE , are in Arithmetical Progression, or as 1. 2. 3. 4. 5, &c. and so do represent the Logarithms; to which the Geometrical Lines Aa, Bb, Cc , &c. do correspond.

26. Let each of the equal Parts ED, DC, CB , &c. be divided equally again in F, G, H , and let the Parallels Ff, Gg , &c. be drawn, and be mean Proportionals between the Collateral ones; that is, $Ee, Ff :: Ff, Dd :: Dd, Gg$, &c. Let there also be more mean Proportionals

onals drawn from the middle of each Subdivision EF, FD, DG, and so on, till these Parallel Lines grow- ing very numerous, have at last but a very small di- stance from each other; then imagine a Curve Line drawn thro' all the Extremities of these Parallels as *e o u f d g h a*: by this means you will gain a Line, whose pro- perties are very considerable and its uses equally great, as shall be shewn in its Proper Place.

27. If this Figure were drawn on a very large Table and with all requisite Exactness; each part A B, B C, &c. might be divided not only into an 100 or 1000, but e- ven into 10000, 100000 equal Parts and more. So that AB being 10000, AC would be 200000, AD 300000, &c. as must always be an Arithmetical Progression.

28. The Line E e being suppos'd to contain 10000 Parts; let us imagine thro' each of those Divisions a Parallel to be drawn to the Line A E cutting the Curve in so many Points, *v. gr.* Let the Line *i o* be drawn thro' the Division 9900 of the Line E e and which cuts the Curve in the Point *o*. Let there be al- so suppos'd the Parallel (to E e) O o, cutting the Line A E in the Division 399553. Then any one may know that 399563 is the Logarithm of the Number 90000. In like manner if S u pass'd thro' the Division 9000 of the Line E e, and the Line *u v* were drawn cutting A E in 395424, then would that Line *u v* be the Logarithm of 9000, &c.

29. So that by this means a Table of Logarithms from 1 to 10,000 may easily be made; and farther, by producing the Line A E.

30. Note, to obtain all the Logarithms from 1 to 10,000; 'twill be enough to seek the Logarithms from 1000 to 10000: That is (having drawn the Parallel *d i*) to take the Logarithms of all the Divisions from *i* to *e*, which Logarithms are all contained between E and D. For by this you will have the Logarithms of all the Parts that are between *i* and E; and whose Logarithms lie between D and A: For Example, since O o is 9900 Parts, and its Logarithm 399563; the same Number may

may be taken for the Logarithm of 990 which is N_n , as also of the Number Y^y 99, changing only the first Figure 3. Because according to the Composition of this Line ON or NY ought to be equal to ED or DC , as any one may easily Prove. So that ON or NY will contain 100, 000 and because AO is 399563, subtracted ON 100, 000, there will remain 299563 for AN . From whence also taking 100, 000, there will rest 199563 for AY . And after the same manner, having AY 395424 for the Logarithm of Vu which is 9000; you may have also 095424 for the Logarithm of Xx which is 9. Or 195424 for the Logarithm of 90, or 295424 for the Logarithm of 900.

3. All this may be reduced to Practice for Calculation, without actually drawing these Figures, but only Imagining them to be drawn. For by the Rules of Common Arithmetick we may find out Ff , the mean Proportional between Dd and Ee , and after that, another Mean between Dd and Ff , or between Ff and Ee &c. But what we have here explained is sufficient to gain as much Knowledge as is necessary for us to have of the Nature and Composition of Logarithms: There being no need for us to undergo the Labour of Calculating Tables of Logarithms, since it's already so well and so often done to our Hands. God, for the Publick Good, having raised some Persons, whom he was pleased to endow with sufficient Patience to surmount so tedious and laborious a Work, as one would think to be insupportable. For we know that above 20 Men were engaged in such a Calculation for above 20 Years together with indefatigable Industry and Assiduity.

(*Par die speaks here a little Coverly, seeming willing to insinuate that this most useful and admirable Work was done first in his own Country; whereas the Logarithms were the Invention of my Lord Neper a Scotch Baron, and the first Tables were Calculated by him, with the Assistance of our Countryman Mr. Henry Briggs.*)

Of late several Improvements have been made in this Matter: As by Nicholas Mercator, of which see Dr. Wallis's

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Wallis's Thoughts in *Philosoph. Transact.* N^o. 38. John Gregory hath given us a way to find *Logarithms* to 25 Places by help of the *Hyperbola*. But Captain Halley in *Philos. Transact.* N^o. 216. shews a way from the bare Consideration of Numbers, and without by the help of Mr. Newton's Way to find the Uncie of the Numbers of a Binomial Power, &c. By which you may find compendiously the *Logarithms* of all Numbers to above 30 places. And he gives there several series for this purpose, some Universal, and some appropriated to a peculiar sort of *Logarithms*.

32. Besides these two kinds of Progressions, there is also a third which is call'd *Harmonical* or *Musical*: Where three Terms being taken which immediately follow one another, it is found that the greatest is to the least :: as the difference between the greatest and the middle one, is to the difference between the middle one and the least, as 30. 20. 15. 12, &c. are in such an *Harmonical* or *Musical* Progression. For taking 30, 20 and 15. the difference between and 30 and 20 is 10, and the difference between 20 and 15 is 5. But 10. 5 :: 30. 15. Therefore these are in *Musical* Progression.

35. This Progression may be diminished Infinitely, but cannot be encreas'd so.

"What is here said of this kind of Progression hath no great use: And I don't design here (*saith Pardie*)

"to medle with uncommon and extraordinary things.

"But hereafter in the further prosecution of these Elements, I shall take notice of some properties of this

"kind of Progression very well worth observing: Because

"cause they will give us some Light into what we

"have of the Musick of the Ancients, about which we

"are yet much in the Dark. There shall be Demon-

"strated the Relation that the *Hyperbola* hath to this

"kind of Progression. For as a *Rectilineal* Angle is

"subservient to the finding out as many *Arithmetical*

"Means as one will, between two given Quantities; and

"as that *Curve Line* which we have above described

"for the *Logarithms*, serves us also to discover as many

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" Geometrical Means as we please between two given Quantities; so it shall then be shewed, that the Hyperbola will help to find out between two given Quantities as many Harmonical Means.

34. There is also a Progression of Squares, Cubes, Biquadrates, Sur-solid's, Quadrato-Cubes, &c. as 1. 4. 9. 16. 25. 36. &c. which are all Squares, whose Roots are the natural Numbers 1. 2. 3. 4. 5. 6. &c. so also 1. 8. 27. 64. 125. 216. are the Cubes of the same Numbers, Also 1. 16. 81. 256. 625. 1296. &c. are the Biquadrates of those Numbers.

35. In a Progression of Squares where 0. is the first Term as 0. 1. 4. 9. 16. &c. The Sum of all the Terms is greater than the third Part of the Product of the last Term multiplied by the Number of all the Terms; And this excess above the third Part, is always so much less, by how much the Number of the Terms is greater.

So in a Progression of Cubes the Sum of all the Terms is greater than the fourth, as before: And in a Progression of Biquadrate Numbers, 'tis greater than the fifth Part, and so on in the rest. To prove which, it will be sufficient to bring an Induction of particulars, as may be seen in this Table where the second Row contains a Progression of Squares from 0. and the third shews you the Sum of the Terms. *v. gr.* You may see here that

1	0	0	0	0	1	or	$\frac{1}{1}$	+	$\frac{1}{1}$
2	1	1	2	2	$\frac{1}{1}$	or	$\frac{1}{1}$	+	$\frac{1}{1}$
3	4	5	12	12	$\frac{1}{1}$	or	$\frac{1}{1}$	+	$\frac{1}{1}$
4	9	14	36	36	$\frac{1}{1}$	or	$\frac{1}{1}$	+	$\frac{1}{1}$
5	16	30	80	80	$\frac{1}{1}$	or	$\frac{1}{1}$	+	$\frac{1}{1}$
6	25	55	150	150	$\frac{1}{1}$	or	$\frac{1}{1}$	+	$\frac{1}{1}$
7	36	91	252	252	$\frac{1}{1}$	or	$\frac{1}{1}$	+	$\frac{1}{1}$

the Sum of the Terms from 0. to 9. is 14. The fourth Rank contains the Product of each Term multiplied by the Number of the Terms from it to 0. which Number is put down in the first Rank; thus 36 is the Product of

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of 9. multiplied by 4. The fifth Rank hath the Fractions exhibiting the Proportions of the Numbers in the third and fourth Rows: As right againſt 14 and 36 you have $\frac{7}{14}$ by which Fraction we would ſhew that 14 is to 36 :: as 7. is to 18. and that the Summ of the Terms, 14. is to the Product of 9. multiplied by 4. viz, 36 :: as 7. is to 18.

Moreover in the ſame fifth Row after $\frac{7}{14}$ you ſee alſo theſe Characters (or $\frac{1}{3} + \frac{1}{11}$) by which we mean that $\frac{7}{14}$ is as much as $\frac{1}{3}$ added to $\frac{1}{11}$. Becauſe in reality $\frac{7}{14}$ is the ſame thing as $\frac{1}{3} + \frac{1}{11}$, that is as $\frac{1}{3} + \frac{1}{11}$. So that the Summ 14 is one third of the Product 36, and over and above contains $\frac{1}{11}$ of it.

So alſo it will be found that 30. which is the Summ of the Terms to 16 is more than the third of 80. which is the Product of 16 by 5. And that the Exceſs is $\frac{1}{4}$; For $\frac{30}{80}$ are $\frac{3}{8}$ or $\frac{3}{4} + \frac{1}{4}$ or laſtly, $\frac{1}{3} + \frac{1}{4}$. But $\frac{1}{4}$ is not ſo great as $\frac{1}{11}$: So that we ſee in the Continuation of this Table, that theſe Exceſſes which are above the third Part of the aforeſaid Product do continually decrease, as the Number of the Terms increaſes: For theſe Exceſſes will be $\frac{1}{14}$, $\frac{1}{14}$, $\frac{1}{16}$, $\frac{1}{16}$, $\frac{1}{18}$, &c. The Denominator of the Fraction always encreaſing by the Number ſix.

36. If ſuch a Table were made for Cubes, it would be found that the Fractions which are the Exceſſes above the fourth Part (as before) would alſo continually decrease in value, their Denominator encreaſing by 4 as often as any new Term ſhould be added to the Progreſſion. And thus it will be found to be in all other Progreſſions (of theſe kinds) if ſuch like Tables be made; as was ſaid in General, in the Precedent Proposition.

All which are of great uſe in the ſubſequent Parts of *Geometry*: Where we ſhall treat alſo of many other Kinds of Progreſſions.

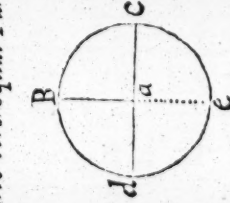
BOOK

BOOK IX.

Problems or Practical Geometry.

That Proposition is call'd a *Problem* in (Geometry) which teaches us how to do any thing, and demonstrates also the Practice of it: Whereas *Theorems* are Speculative Propositions, in which are consider'd the Affections and Properties of Things already done.

2. To divide a Circle into four and into 6, and all Arks into two equal Parts. To divide it into four Parts draw



two Lines as *d a c* and *B a c* at Right angles to each other. To divide it into eight Parts; Biffect the four Arks *B c. c e, &c.* which is done by striking (without the Ark *B c*) two other Arks, with the same opening from the Points *B* and *C*, for if a Line be drawn, from the Point

where those Arks cross each other, to the Centre *a*, it shall Biffect the Ark *B C*. The like is to be done for the other Arks.

To divide a Circle into six equal parts; you need only take the length of the Radius: and applying it six times about the Circle, it will exactly divide the Circumference into six equal Parts, and thus by a new Biffection, may a Circle be divided into 12, 24, 48, or into any Number of equal Parts, &c.

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3. To divide a Circle into five, into fifteen, and into other Equal Parts. This may be done thus; (as I demonstrate in *Algebra*). Make a Rectangle Triangle, one of whose Legs shall be the Radius of the Circle and the other half the Radius. From the Hypothenufe of this Triangle, take half the Radius; the remainder shall be the Chord of 36 deg. and the side of a Decagon. Double that Ark, you have the Ark of 72 deg. (*whose Chord is the side of a Pentagon*) and it is the fifth part of the Circumference; and the said Chord shall be also the Hypothenufe of a Rectangle Triangle, one of whose sides is the Radius and the other the side of a Decagon. And as by the last was found the Chord of 60 deg. so by Subtracting the Chord of 36 deg. from 60 deg. you may have the Chord of 24 deg. which is the 15th part of the Circumference. But for Practice, the shortest and surest way, is by repeated Trials with the Compasses to find a Distance that will go precisely five times about the Circle: Then divide, after the same manner (by Trials) that distance into three equal parts exactly: So shall you gain a Chord that will divide the Circumference into 15 equal Parts, and then dividing each of those 15 Chords into four equal Parts, and each of those into six; you will divide the whole Circumference into 360 deg. And this Division is most Commodious for Practice and Use. Note, that the way to divide a Circle into 3, 5, 7, or into any other odd Number of Parts is not yet found Geometrically; Geometrically I say, that is, by making Use only of a straight Line and a Circle.

" This Division of a Circle into 360 deg. is very useful, when a Person understands how to use the *Compasses of Proportion* (or *Sector*.) This is called because 'tis a kind of Compasses with broad Legs: As *a B*. *a C*. on which are described divers Lines and Divisions; But those which are most in use, are of two sorts



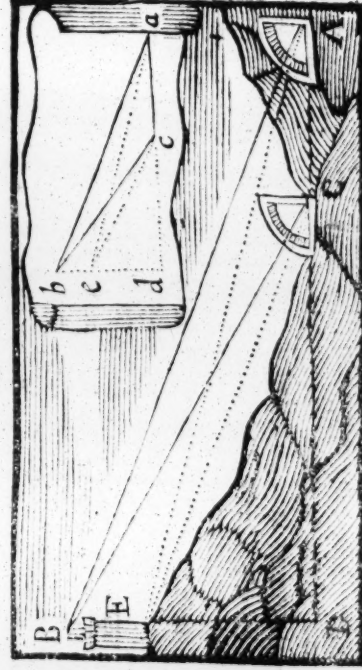
“forts. On one side of this Sector, and on each Leg, “is a Line aeB and aeC , which serves to divide a “Circle into 360 deg. at once, and also to take at any “time as many Degrees as you please. And this Line “on the Sector is thus divided.

4. *To divide and graduate the Sector, that it may serve for the Division of a Circle.* Imagine a Semicircle aEB accurately divided into 180 deg. if then from the Point a , as from a Centre, you transfer the Divisions of the Semicircle into a Line aB . *v. gr.* If from E , 60 deg. you draw the Ark Ee , and if from D 90 deg. in the Semicircle, you draw the Ark Dd , &c. Then ought 60 deg. on that Leg of the Sector, to be placed at the Point e , and 90 deg. at the Point d , &c. And if you Transfer the same Degrees after the same manner into the other Leg aC , you will graduate the Lines aB and aC (on the Sector) as they ought to be for this purpose and they will be two Similar Lines of Chords.

5. *To explain the Use of the Sector as far it serves for the Division of a Circle.* Let there be a Circle given Af . take with your Compasses the Radius Af and (keeping that distance) set one Foot of them in e or 60 deg. on one Leg of the Sector; move the other Leg of the Sector to and fro so long, till the other Point of the Compasses falls exactly on e or 60 deg. in that Leg of the Sector: So that the Distance ee be exactly equal to the Radius Af . Then if you would have readily 90 deg. of that Circle; (Letting the Sector lye still, and always keeping the same Angle). Open your Compasses till the Points fall exactly on d and d or 90 deg. and 90 deg. on each Leg of the Sector: And then that Distance transferr'd into the Circle, in $f g$, gives you the Ark of 90 deg. *fg*. So also if you would have had 35 deg. you need only apply your Compasses to 35 deg. and 35 deg. on each Leg of the Sector in the Lines (of Chords) aB and aC ; and that Distance transferr'd into the Circle, shall cut of the Ark of 35 deg. and thus may you proceed to find any Degrees you please. All which is grounded on the 42 43, 49, and 50 Propositions of the

30 deg. from e to f ; then thro' f draw the Line af , which with the Line ac will make an Angle of 30 deg.

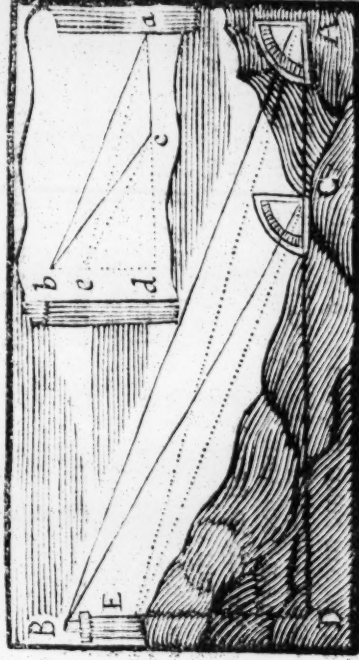
8. *Having the Angles of any Triangle and one side given to find the other two sides.* Suppose you are told there is a certain Triangle some where, whose Base AC is 10 Fathom; and that the two Angles at the Base are ACB , 150 deg. and CAB 20 deg. (and consequently the remaining Angle at the Vertex or Top must be 10 deg. for the Sum of 150, 20 and 10, is just 180 deg. which is two Right Angles). You are required to tell how many Fathom there are in the other sides AB and AC . Make on Paper, or rather on fine Pastboard, a Triangle abc Similar to the propos'd one after this manner. Take a Bale at pleasure ac and from any Scale of equal Parts let it be



10 Inches, half Inches, &c. in Length. On this Line ac make two Angles, one cab of 20 deg. and the other acb of 150 (9. 17). Then will the two Lines ac and cb cross one another when produced in the Point b . Then measure (on the same Scale you took the Base ac from) how many Inches, &c. the Lines ab and cb are in Length; And you may be assured that there are just so many Fathom in the Lines AB and CB sought, as you find Inches, &c. or any equal Parts, in the Lines ab and cb . For since the Triangles are Equiangular. they are Similar and therefore $ac. ab :: AC. AB$, &c.

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9. To Measure Distances, Heights, Depths, and in Dimensions of all Remote and General, the Magnitudes and Inaccessable Places. If on the top of any Hill appearing at a distance there were a Tower as B E and its distance from us and its Height were required: You must first with some Instrument (as with a *Quadrant*, that is the fourth Part of a Circle divided into 90 *deg.* and furnish'd with a Ruler, or Label with Sights, and moveable on the Centre) you must I say with some such Instrument, take two Angles at two several Stations in this manner: If you are in the Station A, place your Instrument so, that one side of it may answer exactly to the Horizontal Line A D; and keep it without raising or depressing it in this Position. Then place your Eye at A, (that is at the Centre of the Instrument), and turn the Label till it Point to the Top of the Tower B, and that looking thro'



the Sights you can see the top of the Tower exactly; then will the Label cut in the Limb of the Quadrant the Degrees of the Angle B A D, for the Limb is suppos'd to be graduated for this purpose, Then change your Station, moving in a Right-line forwards 10 Fathom (or it might have been any other Distance, and backward as well as forward) to C. and there take after the same manner the Angle B C D: By which means you will have also the Angle B C A, because those two together make 180 *deg.* or two Right ones. So that in the

Triangle ACB you have now found the Base AC which is 10 Fathom, and also the two Angles at the Base; and consequently the sides CB and AB may be known (9. 18). And then you may have the Height DB or the distance AD , if you make a little Triangle Similar to that, and there from the Point b let fall a Perpendicular bd to the Base Line AC continued to d . For BD or AD will be just as many Fathoms as bd and ad will be equal Parts measur'd on the Scale, (as in the last.) And if after you have thus gain'd the Height BD , you find, by the same method, the Height ED also, you may (by *Subtracting this Altitude from the Former*) find the Height of the Tower EB .

NB. (*The common Quadrant with a String and Plummet and with the Sights fixt on one of its sides, is more convenient and ready than this of Pardies, which is now out of Use*).

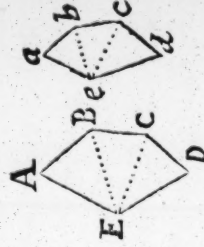
“Some times instead of advancing towards the Tower
“and of making Observations of the Height below;
“or of those Angles the visual Rays make with the Horizontal Line, it is convenient to take two Stations
“side ways of each other; But it comes all to the same
“and the Practices in reality are not at all different.

“And by this means, as any one may see, may all
“imaginable Heights and Distances and other Dimensions be taken; provided we can but come to observe
“their Extremities, from two different Places. I shall
“not stay now to describe the Particular ways of doing
“this; nor to enumerate the great Advantages that
“would accrue from the use of Teleſcopical Sights
“fixt on the Label, or on the side of the Instrument
“used in taking Angles; which indeed is an Invention of
“ineſtimable Benefit to Surveyors.

10. *To take the Plane of any Place.* Let $ABCDE$ be a City, or any other Place, and you were required to take the Plane, and to make a Draught of it. Take all the Lengths of its sides, and of Lines drawn from
Angle

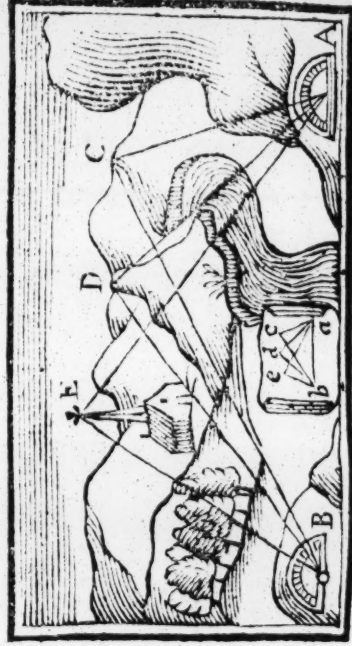
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Angle to Angle: And transfer all these upon Paper, laying them down according to their true Proportion. For Instance, having found AB to be 30 Paces, BC to be 59, CD to be 50, BE to be 67, and AE 49, &c. and having ready drawn on Paper a plain Scale divided into 100 equal Parts. Make the Line *ab* 30 of such Parts; *be*, 67; and *ac*, 49, then those Lines drawn and join'd together will make the Triangle *abe* every way Similar to the Triangle ABE. And if you go on thus and make the Triangle *bec* Similar to BEC, &c. you will form the Figure *abcde* every way Similar to the Plane of the Place ABCDE.



11. But if you cannot get into the Place to Survey it, and to Measure the distance between the Angles EB and EC, you must take the several Angles of the Plane, and transfer them into your Draught; so that if the Angle BAE be 66 deg. the Angle *bae* must also be 66 deg. and so of all the rest.

12. To make a draught of any City or Country. Ascend up into any two elevated Places from whence you can



plainly see the City or Country whose Delineation you would make. And having with you a Quadrant, whole

Circle, or Semicircle well divided into Degrees, together with its Label (with Sight) upon its Centre: Place your Instrument at A, and so that one of its sides may lye in a Line between A and B, which done, and the Instrument fixt there, observe the several Steeples, eminent Houses. Towers, Hills and all other remarkable Places as E D C, &c. and take their Angles with the Label and Sight, and write them all down to help your Memory. Thus, let the Angle C A B be 50 *deg.* 30 *min.* the Angle D A B 45 *deg.* 8 *min.* &c. Proceed

after the same manner at the Station B; noting down the Angle A B C to be 40 *deg.* 10 *min.* the Angle A B D 47 *deg.* 28 *min.* &c. After which draw on Paper any Line at Pleasure as *a b* and make at each end of it Angles equal to those which you found *c a b* equal C A B, *d a b* equal to D A B, and *a b c* equal to A B C, &c. and by this means you will have the Points *c*, *d* and *e*, &c. which will be in the same Position to one another as the Steeples, or other eminent Places C D E, &c. are. And thus having drawn the most Conspicuous and Principal Places the rest may easily be taken by the Eye. But to make this Operation very exact, 'tis convenient to take the Angles also at a third or fourth Station, and then if they all agree, any one will know that the Work was well done.

13. Having the two sides of a Triangle and the Angle between them given, to find the third Side and the two other Angles

14. Having two sides, and an Angle opposite to one of them given to find the Side and the remaining Angle

15. All the Angles and one side given to find the rest

16. The three sides being given to find the Angles

All which may easily be found by drawing on Paper a Triangle Similar to that whose Sides or Angles are in part sought.

17. To measure the Area, (that is) the Capacity of, or Space contain'd in any Right-line Triangle given *a b c*, From the Top *b* let fall a Perpendicular *b d* to the Base *a c*, (produced if there be an occasion). Then divide *a c* into

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into 10, (or any other equal Parts you please) and find how many such parts are contained in bd , for then multiplying the half of bd by 10, you will have the Area of the Triangle (3. 18), as if db contains four such parts of which ac contains 10. You multiply 10 by 2 (half of 4) and the Product will be 20: which is the Area of the Triangle abc . That is, the Triangle contains as much space, as 20 little Squares do, whose Root or Side is the 10th part of the Line ac .



With Regard to Practice, there is no method easier nor exacter than this: (Only there is no need of actually Dividing ac into 10 or any other number of any Parts as above: For having a good Diagonal Scale, you need only measure there the length of the Base and Perpendicular: and then multiplying one by the other half the Product is the Area; or half one, Multiplied by the other, is the Area).

But in some certain Cases 'tis necessary to know how to Measure these things to such a Degree of Exactness as cannot be attained, but by means of Calculation: And therefore I shall here give you the Principles, on which all that Art depends.

18. In a Right angle Triangle abd , two sides being given to find the Third, by Calculation.

Let the Leg bd be 3 Fathom, (See the last Figure) and the Leg ad 4 Fathom, multiply 3 by 3, and 4 by 4, and you will produce the two Squares, 9 and 16. The sum of which two Squares is equal to the Square of the Hypotenuse ab : (2. 61.) and consequently the Square of ab is 9 added to 16, i. e. 25. And therefore to find ab I need only extract the side or square Root of 25 which is 5: and I conclude ab to be 5 Fathom.

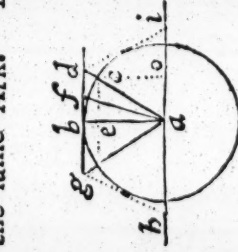
If the Hypotenuse ab 5 be given, with the side ad 4, you must then subtract the Square 16 from the Square 25, there will remain 9, whose Square Root 3, is the Leg db sought. But it often happens that the

two

two Squares of the Legs added together, do not make a square Number: and also as often, the square of the one Leg substracted from the square of the Hypothenuse doth not leave a square Number: as if the Legs had been 2 and 3, their squares had been 4 and 9; whole sum is 13. But 13 is not a square number, and consequently can have no exact Root, nevertheless there are numbers which will come near it. As $3\frac{1}{3}$ is pretty near the Root of 13, for $3\frac{1}{3}$ multiplied by it self, makes 13 within $\frac{1}{3}$ part: so that the side ab , in this case, would be $3\frac{1}{3}$ and a little more.

I don't shew here the way of Extracting the square Root: because 'tis a Rule in a Arithmetick, of which I don't, Treat here.

19. To Calculate the Tangent, Secant, and Sine of 30 degrees. Let $b a$ be the Radius or whole Sine, $a d$ the Secant of 30 degrees, $b d$ the Tangent, and c the line of the same Ark. 'Tis easie to see that $b d$, here is



the half of ad : For drawing ag another Secant of 30 degrees, the Triangle agd will be equilateral, because each Angle g, d , and gad will be 60 deg. wherefore bd being half of gd must also be the half of ad . And for the same Reasons ce will be the half of ac . Supposing then in the Rectangle Triangle ace the Hypothenuse ac be 2, and the side ce 1, therefore taking the square 1, from the square 4, there remains 3, which is equal to the square of the side ae , which is equal to the line of the Ark ci , or 60 degrees.

But now if instead of taking 2 and 1, for ac and ce , you take, 1000000 and 500000, and substract the latter from the former, there will remain 750,000, 000,000 whose square Root is very near 866025, for ae or oc , the sine of 60 degr.

20. Hence knowing ec the Right Sine of any Angle, co the Sine Complement, or Co-sine of that Angle is known. The Complement of any Angle or Ark is what it wants of

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of 90 deg. thus having the Angle $c a b$, 30 deg, it^s Complement $c a i$, 60 deg. is also known by Substracting 30 from 90. And its Sine is found by the precedent.

21. *Knowing c the Sine of any Angle and $c o$ or a its Sine Complement to find the Tangent $b d$ and, the Secant $a d$.* Since the Triangles $a e c$ and $a b d$ are Similar, it follows, that, $a e . e c :: a b . b d$. and consequently the Tangent is found; Also, as $a e . a c :: a b . a d$. and therefore the Secant is found, and the Ark $c b$ being 30 deg. the Tangent $b d$ will be 577, 350 and the Secant $a d$ 1, 154, 700 such parts, as the Radius is supposed to be divided into.

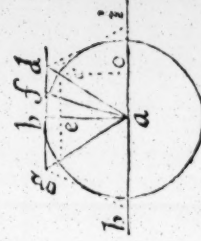
22. *Knowing the Sine, Tangent and Secant of any Ark $b c$, to find the Sine, Tangent and Secant of half that Ark.* Bisecting the Ark $b c$, draw $a f$ from the middle Point to the Centre and then $d f . f b :: a d . a b$, (6. 72). and consequently $b f$ the Tangent of 15 deg. is found, as also the Sine and Secant of the same Ark of 15 deg. After which by Bisecting the Ark $b f$, you may, by the same way, find the Tangent Sine and Secant of 7 deg. 30 min. and after that of 3 deg. 45 min. and so on as you please.

23. *To find $c e$ the Sine of 45 deg.* This is equal to the Co-sine of the same Ark of 45. that is to $e a$ and consequently the Tangent and Secant of that Ark is also found, and (by the last) of its half 22 deg. 30 min. and of the half of that 11 deg. 15. min. &c.

24. *To find the Sine of 36 deg.* Inscribe a Regular Pentagon in the Circle, the Proportion of whose side to the Radius is known (9. 13). But that side is the Chord of 72 deg. and therefore its half will be the sine of half 72 deg. that is of 36 deg. And the sine of the same being thus known, the Tangent and Secant of the same Ark are known also: And also the Sine, Tangent and Secant of the half of 36 deg. which is the sine of 18 deg. And then of 9 deg. 4 deg. 30 min. 2 deg. 15 min. &c.

25. *To find the Sine, Tangent and Secant of 12 deg. and of the halves 6 deg. 3 deg 1. deg. 30 min. 45 min. &c.*

This



This is done by finding the Chord of 24 deg. which is the side of a regular Polygon of 15 sides (9. 13). for its half is the fine of 12 deg. &c.

26. And Collecting all these together we shall have the Sines, Tangent and Secants of the Angles of 45 min. of 1 deg. 30 min. of 2 deg. 15 min. of 3 deg. 45 min. of 30 min. and so of all the rest from 45 min. to 45 min.

27. To find the Sines, Tangents and Secants of all Arks that are between the two Arks thus found from 45 min. to 45 min. This is done by the Rule of Proportion: For instance, the natural line of 45 min. being 1308, the fine of 1 min. will be 29. because as 45 min. 1 $\text{min.}::$ 1308. 29. And thus the fine of 20 min. will be 581.

So also to gain the fine of 3 deg. 30 min. you must Work thus; having the Sine of 3 deg. 5233. (9. 35) as also the fine of 3 deg. 45 min. which is 6540. (9. 32). 'tis manifest that these 45 min. which are between 3 deg. and 3 deg. 45 min. have for their fine 1307: for Subtracting 5233 the fine of 3 deg. from 6540 the fine of 3 deg. 45 min. There remains 1307. If therefore you would have the fine of 3 deg. 30 min. you must say,

If 45 min. (the difference between 3 deg. and 3 deg. 45 min.) have for their fine 1307; what shall 30 min. have? (which are the difference between 3 deg. and 3 deg. 30 min.) the answer will be (working by the Golden Rule) 873. which if added to 5233 will make 6104 the fine of 3 deg. 30 min. And so of all others.

And by this means may Tables of (Natural) Sines, Tangents and Secants be made, from one Minute to 90 deg.

Observe that by the last Rule the sines cannot be found exactly, because the sines do not encrease in the same Proportion as their Arks do; but the difference is so very small that you need not Trouble your self to be more exact.

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28. By the help of such Tables as these, Triangles may be Calculated; because 'tis certain that in all Triangles, the sides are to one another as the sines of the opposite Angles. For Example, In the Triangle abc , a Circle being drawn which shall Circumscribe it, the Perpendiculars ei and eb drawn from the Center to the Sides shall Bisect ab and bc (4. 6) so that $ab :: ai . bh$. But ai is the Sine of the Angle aei or its equal acb (4. 13) and bh also is the sine of the Angle feb or of bac which is equal to it. Wherefore the sides are as the sines of the opposite Angles. Q. E. D.



29. And upon this Principle, *Knowing one Angle and two Sides, or one Side and two Angles you may find the rest.* Work by the *Golden Rule* and say, as one side known is to the sine of the Opposite known Angle :: So is the other known side, to a fourth Number: which will be the Angle opposite to that other side.

Or if two Angles and one side are given, you must Work thus, as the sine of one given Angle, is to the known side opposite to it :: So is the sine of the other known Angle to a fourth Term: which will be the side opposite to that other Angle, &c.

(N.B. By this Axiom, as 'tis called in Trigonometry, *that the Sides are as the sines of the opposite Angles*; you can only solve the three first Cases of Oblique Triangles; which our Author should have Intimated. For when *two Sides* and the *Angle between them* in an oblique Triangle are given. and either the *Opposite Side*, or the *other two Angles* are requir'd: You must Work by this Axiom, and first find the other Angles thus:

“As the Sum of the Legs about the Angle given, is to their Difference :: So is the Tangent of half the Sum of the other two Angles, to the Tangent of half their difference.”

“Now

“ Now the Sum of the other two Angles is known,
 “ being what the given Angle wants of 180 *deg.* and
 “ their difference is now found, add therefore their
 “ half Sum, and half difference together and it gives
 “ you the *greater of the two Angles sought*; and half the
 “ difference Subtracted from the half Sum leaves the
 “ lesser Angle sought. And thus having found the An-
 “ gles; if the side opposite to the former given Angle
 “ be sought, it will be found easily, by *Pardies Axiom*,
 “ that the sides are as the sines of the Angles.

The Demonstration of which Axiom is briefly thus,

D E M O N S T R A T I O N.

I say the Summ of the Legs of any Angle a, is to their Difference :: as the Tangent of $\frac{1}{2}$ the Summ of their opposite Angles, is to the Tangent of half their Difference.

Produce O one of the given Legs of the Angle given, till *af* become equal to H or C*a*, and then Bisect *bf* in *e*, join *cf*, and Bisect it also in *d*: draw *ad*, which will be Perpendicular to *cf* (2. 16) and draw *de*, which will be Parallel to *cb*: (6. 92) Then will the Angle *cad* = *daf*: i. e. to the half of *caf*. which External Angle *caf* = *c* + *b*, That is, to the Summ of the opposite Angles required.

Draw then *ga* Parallel to *cb*: So will the Angle *gac* be equal to the alternate one *c*. And if from $\frac{1}{2}$ the Summ of the opposite Angles you take the lesser Angle: i. e. if from *cad* you take *gac*, there will remain the Angle *gad* equal to half the Difference of the opposite Angles.

And so also, if from *be* half the Summ of the Legs, you take O the lesser Leg, there will remain *ae* equal to half the Difference of the Legs. And then since the Triangle *cad* is Right-angled, (by what is proved below in the next Figure save one) if *ad* be made Radius, *cd* will be the Tangent of the Angle *cad* (i. e. the

First, Given H , O and a , required c , b .

For $H + O - H - O ::$ As T , $\frac{1}{2} Z$ opposite Angles, is to T , $\frac{1}{2} X$ opposite Angles, and then $\frac{1}{2} Z + \frac{1}{2} X = b$ and $\frac{1}{2} Z - \frac{1}{2} X = c$.

Second, Given H , O and a , required B ?

First find the Angles by the former Case, and then $S, b, H :: S, a, B$; or $S, c, O :: S, a, B$. by *Pardies's* General Axiom.

"There is also another Case in plain oblique Triangles which requires a particular Axiom to solve it; and that is, where *All three sides are given to find the Angles*. Here, let fall a Perpendicular from any An-

gle to its opposite Side as ap .

"And then say, as the side dc

"is to $da + ac$ the Summ of

"the other two Sides :: So is

"the difference of those two

"Sides $da - ac$ to a fourth

"Number. Half of which

"added to $\frac{1}{2} dc$, gives you

"the Segment of the Base dp ;

"and if Subtracted from $\frac{1}{2} dc$, it will leave the other

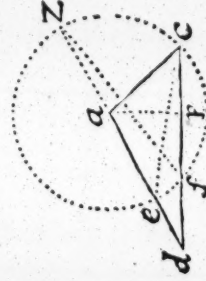
"Segment pc . And when those Segments are thus

"found, the Angles are easily had thus: $d a. Radius ::$

" dp . Co-line of the Angle d . And $a c. Radius :: pc$,

"Co-line of the Angle C .

The Demonstration of which last Axiom is this,



D E M O N S T R A T I O N.

On the Center, with the Distance ac describe a Circle, which will Intersect both the other Sides of the Triangle, and then pd will represent the Summ of the Legs aa and ac : ae will represent their Difference, and

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and *cf* will represent the Difference of the Segments of the Base made by the fall of the Perpendicular *aP*.

Then I say (by 6. 67) $dc \cdot dz :: de \cdot df$. That is, *The Base is to the Summ of the Legs :: as the Difference between the two Sides is to the Difference of the Segments of the Base*; as is apparent from that Proposition, after drawing the Perick Lines *ec* and *fz*.

And then the Case will stand thus,

Third, Given *H, O, B*, all the Three Sides, required the Angles. See the Figure before.

I say by this Axiom $B \cdot H + O :: H - O \cdot X$ which expresses the Difference of the Segments of the Base, $= df$ in the Figure.

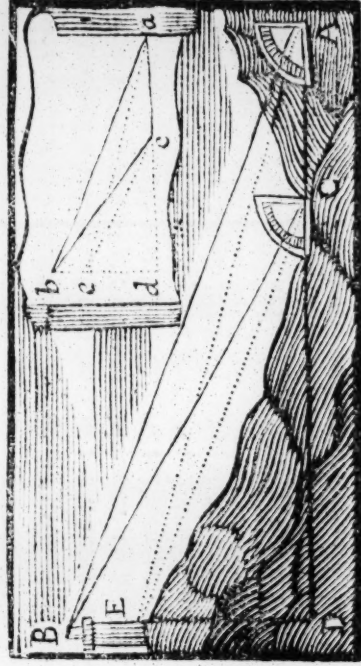
And then having found $X \cdot \frac{1}{2} B + \frac{1}{2} X = dp$, the greater Segment, and $\frac{1}{2} B - \frac{1}{2} X = pc$ the lesser, by which means the two Right-angled Triangles *adp* and *apc* will be solved easily: For *d a. Radius :: dp. Co-sine of the Angle d. and a c. Radius :: as pc. to the Co-sine of the Angle c*: As above said.

“ And thus you have here a Short Compendium of all
 “ plain Trigonometry, only *Parodie* hath not particularly
 “ insisted on the Solution of Right-angled Triangles a-
 “ part; But tho’ it be true that these Axioms here men-
 “ tioned will help us to solve all plain Triangles what-
 “ soever, whether Right or Oblique; yet ’tis more con-
 “ venient to consider Rectangled ones, as a distinct
 “ Species by themselves, as is usually done in all Trigo-
 “ nometrical Books: For in them the Right-angle being
 “ always known, you need have but two Terms given
 “ you, to find the rest. And for the Solution of all the
 “ seven Cases of Rectangled Triangles you need only
 “ Consider, that which Side soever you make or suppose
 “ to be the Radius, the other two will be either Sines,
 “ Tangents or Secants. If the Hypothense be made Ra-
 “ dius, the Legs will be the Sines of the opposite Angles;
 “ if either of the Legs be made Radius, the Hypothe-
 “ nuse will be the Secant, and the other Leg the Tangent
 “ of

K

“it self consider’d as a Length, at first sought or required. That is R.B.: T, a. P. And therefore the Length of the Perpendicular P will be known. This fourth Case of Right-angled Triangles I chose out from among the rest because ’tis the most difficult of any of them; and therefore this being explained, as I hope it is here, there will be no difficulty in solving any of the other Cases. By this Instance and Explication also you may get a rational and clear Notion of all Trigonometrical Calculation: Wherefore I doubt not of the Reader’s Pardon for this long Digression. I return to our Author.

30. These Operations are very much shortned by the Use of the Logarithms; And therefore care hath been taken not only to make Tables of Natural Sines, Tangents, &c. but also to Calculate the Logarithms of them, which you will find in Books Printed right against the Natural ones. So that instead of Multiplication and Division, which, with intolerable Labour, must be used,



if you Calculate by the Natural Sines and Tangents; you need now only use Addition and Subtraction when you work by the Logarithms. As if in the Triangle ABC, whose Side AC is known, and is 10 Fathom; the Angle ABC is given, and is 10 deg. and the Angle CAB, of 20 deg. is given also, the Side BC be required: You must say as the Sine of the Angle B, (which in the Natural

tural Sines is 17364) is to the Side AC (10 Fathom) ; so is the Sine of the Angle A (which in the Tables is, 34202) to the Side sought, CB. To find which fourth Number CB by the *Rule of three*, you must multiply the second Term 10, by the third Term 34202, and divide the Product 342020, by the first Term 17364. which would be very tedious. But if instead of these Natural Numbers you had used the Logarithms, you need only have added the Logarithm of 20 *deg.* to the Logarithm of 10 Fathom, and from their Sum subtracted

Sin. Ang. A 20 <i>deg.</i>	9.5340517
Logarith. A C.	1.0000000
Their Summ.	10.5340517
The Sin. An. B 10 <i>deg.</i>	9.2396702
There Remains	
That is $19\frac{1}{7}$ Fathom.	1.2943815

the Logarithm of 10 *deg.* there would have remained the Logarithm 1.2943815 which in the Tables will be found to belong to the absolute Number 19 ; so that you may conclude the side BC is 19 Fathom and something more because the Logarithm found is greater than the proper Logarithm of 19 in the Tables. See the Work above.

Books that Treat of Sines and Logarithms do explain this matter more particularly and fully ; but I suppose I have said enough here to make any one able to understand Calculation, without the help of a Master : I shall also in the subsequent Books of Geometry give you some more Propositions on this Subject.

31. To find a Right-Line that shall be very nearly equal to the Circumference of a Circle. Taking the Tangent of 30 *deg.* which is *bd* twelve times, and disposing these Tangents so round about the Circle as that they be joined (in a Right-line) two and two together as you see

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fee in the Figure: Where dg is composed of two Tangents so joined each of 30 min. and so is gb and $d i$, &c. By this means you will make a Polygon of six Sides whose Perimeter or Circumference is greater than that of the Circle (4. 27). And if you take

the Sine ec twelve times, proceeding as before with the Tangents, you will have an inscribed Polygon of six Sides whose Circumference is less than that of the Circle. So that taking the Radius ab 1, 000, 000; bd which

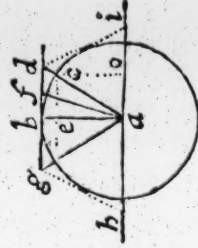
is 11, 577, 350, taken twelve times, that is to say, 6, 928200 is greater than the Circumference of the Circle. And ec which is 5000, 000, taken twelve times, that is, 6000000, is less than the Circumference of the Circle.

32. But if instead of taking twelve times the Tangent and Sine of 30 deg. you take 360 times the Tangent and Sine of one Degree: That is 17455 and 17452 you will make two Polygons, the one Circumscribed, which will be 6283800 and greater than the Circle, the other Inscribed 6282720, which will be less than the Circle.

33. Again, let the Radius 1000000000 be given, taking the Tangent and Sine of one Minute 216000 times (which is the Number of Minutes in 360 deg.) you will have 628318532'00 a little greater, and 628318512000 a little less than the Circle; for the Tangent of one Minute is 29088821, and the Sine of one Minute is 29088820. And if these three Numbers the Radius, the Circumscribed Polygon and the Inscribed Polygon, be divided by 1000000: There will remain for the Radius 1000000. And the Perimeter of the Circumscribed Polygon will be 6, 283185 $\frac{316}{1000000}$ and the Perimeter of the Inscribed Polygon will be 6283185 $\frac{12}{1000000}$. So that these two Perimeters, the one greater and the other less than the Circle, will differ but by one hundred thousandth Part of the Radius. And if you had taken the Tangent and Sine of one second, you might have come

K 3

yet



yet vastly nearer to an Equality between the Circumscribed and Inscribed Polygons.

34. For Practice, 'tis usually supposed that the Diameter is to the Circumference very near as 7 to 22. That is, if the Radius be 7, the Circumference will nearly be 44 such Parts, and this agrees to what we have here delivered; for $7 \cdot 44 :: 100 \cdot 6284$.

35. To find the Area of the Circle. If the Radius be supposed 1000, the Circumference will be very nearly 6283; half of which, viz. 3141 being multiplied by the Radius 1000, Produces 3141000 for the Area of the Circle (4. 31). But if the Radius be taken in any other Measure, as suppose 9 Inches; you must say as 1000. 3141 :: 9. $\frac{3162}{1000}$. And then Multiplying this last Number 16 $\frac{3162}{1000}$ which is the Semicircumference, by 9, the Radius, you will have 173 $\frac{422}{1000}$ for the Area of the Circle to that Measure.

And in my Judgement 'tis better to use this Proportion of 1000 to 3141 than the Common one of 7 to 22.

36. To find the Solid Content of a Parallelopiped or of a Cylinder. Multiply the Base by the Perpendicular Altitude.

37. To find the Content of a Pyramid or Cone. Multiply the Base by one third of the Perpendicular height.

38. To find the Content of a Sphere. Multiply the Surface by one third of the Radius; or a great Circle by $\frac{2}{3}$ of the Diameter.

F I N I S.

A

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Circle

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7. 4

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M *Mean Proportional*

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Measuring of all Heights and distances

Minutes

Mixt Angle

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N.

N *Numbers Plane*
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Cubick

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Square

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Superficies

Or Plane Surface

Sur-solid

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 1. 3
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T *Ables of Natural Sines, Tangents, &c. how made*
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The

*The Characters, Marks, Signs or
Symbols here used, are only these,*

$=$ Qual to.

$+$ More or adding.

$-$ Less or Subtracting.

$::$ The Mark of four Quantities being
discretely Proportional Geometrically.

$\div$ The Mark for Continual Proportion,
or Geometrick Progression.

$:::$ The Mark for Arithmetical Proportion.

or

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ally.
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